Insights into the QCD Phase Diagram from Nambu–Jona-Lasinio Type Models

Mahammad Sabir Ali

National Institute of Science Education and Research

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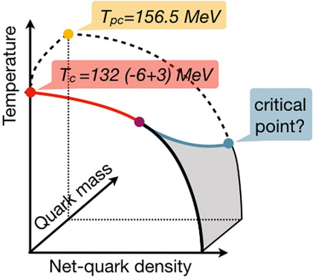
Introduction

- With two massless quarks, QCD has global chiral symmetry

$$SU(2)_V \times SU(2)_A \times U(1)_V$$

- Due to the non-abelian nature of QCD, the interaction strength increases at low energy, resulting in the spontaneous breaking of the chiral symmetry.

Phase Diagram



- Residual symmetry: $SU(2)_V \times U(1)_V$
- With physical quark masses, the crossover temperature is $T_{co} = 156.5$ MeV

QCD at low energy

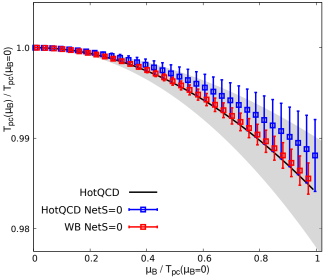
- At low energies, the interaction strength becomes very large due to the non-abelian nature. This forbids one to do analytical calculations.
- To study this low-energy regime, one can simulate QCD on the lattice, which is costly and time-consuming.
- Lattice QDC suffers a *sign* problem while dealing with nonzero chemical potential.
- Simplified effective QCD models governed by QCD symmetries are one of the most used analytical methods to explore the low energy regime.

$T - \mu_B$ phase diagram from LQCD

- To compare between different methods, one parametrized the phase line as

$$\frac{T_{pc}(\mu_X)}{T_{pc}(0)} = 1 - \kappa_2^X \left(\frac{\mu_X}{T_{pc}(0)}\right)^2 - \kappa_4^X \left(\frac{\mu_X}{T_{pc}(0)}\right)^4 .$$

- The chiral transition line in the $T - \mu_B$ plane calculated in LQCD[A. Bazavov et al. (HotQCD), 2019; S. Borsanyi et. al., 2020]



2-flavor NJL model

- The NJL Lagrangian [Y. Nambu and G. Jona-Lasinio, 1961; M. Frank, M Buballa, M. Oertel, 2003]

$$\begin{aligned}\mathcal{L}_{\mathrm{NJL}} &= \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}} \\ &= \bar{\psi}\left(i\partial\!\!\!/ + \mu\gamma^0 - m\right)\psi + G\left\{(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right\}\end{aligned}$$

- In the chiral limit ($m = 0$), the Lagrangian $\mathcal{L}_{NJL}$ is symmetric under

$$SU(2)_V \times SU(2)_A \times U(1)_V$$

- Within mean-field approximation, the Lagrangian

$$\mathcal{L}_{\mathrm{MFA}} = \bar{\psi}\left(i\partial\!\!\!/ + \mu\gamma^0 - M\right)\psi - G\langle\bar{\psi}\psi\rangle^2$$

  with $M = m - 2G\langle\bar{\psi}\psi\rangle = m - 2G\sigma$
- In the symmetry-broken scenario $\langle\bar{\psi}\psi\rangle$ acquire nonzero value.

Free Energy

- Wick rotation

$$t \to -ix_4 \text{ and } \gamma_0 \to i\gamma_4 \implies q_0 \to iq_4$$

- Matsubara formalism to introduce temperature

$$\int \frac{dq_4}{2\pi} f(q_4) = T \sum_{n=-\infty}^{\infty} f(\omega_n)$$

with $\omega_n = (2n+1)\pi T$ for fermions.

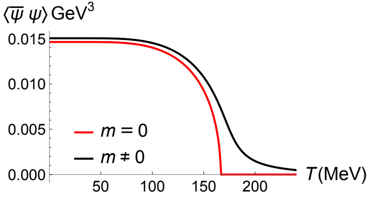
- The free energy becomes

$$\Omega = G\sigma^2 - \mathcal{N} \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} - \mathcal{N} T \int \frac{d^3q}{(2\pi)^3} \left\{ \ln\left[1 + e^{-(E_p-\mu)/T}\right] + \ln\left[1 + e^{-(E_p+\mu)/T}\right] \right\}$$

Phase transition and crossover

- In the chiral limit, this symmetry breaking and restoration at high temperature is connected via second-order phase transition.
- At the transition temperature (T_0), the condensate ($\langle\bar{\psi}\psi\rangle$) becomes zero.

- With nonzero current quark masses, this second-order phase transition becomes a crossover.
- And the crossover temperature (T_{CO}) is the temperature at which the rate of change of condensate is maximum.



Curvature in Chiral Limit (2 Flavor)

- With $m = 0$, the grand canonical potential for a 2-flavor NJL model can be expanded as

$$\Omega(\sigma, T, \mu) = \alpha_0(T,\mu) + \alpha_2(T,\mu)\,\sigma^2 + \alpha_4(T,\mu)\,\sigma^4 + \alpha_6(T,\mu)\,\sigma^6 + \ ...$$

- The second-order phase transition line can be obtained by [S. Gupta and R. Sharma, 2018]

$$\alpha_2(T,\mu) = -G' + \frac{T^2}{8\pi^2}\int p^2\,dp\left[\frac{1}{(1+e^{(p-\tilde{\mu})})p} + \frac{1}{(1+e^{(p+\tilde{\mu})})p}\right]$$
$$= -G' + \frac{1}{48}\left(T^2 + \frac{3}{\pi^2}\mu^2\right) = 0.$$

- The curvature coefficients [S. Gupta and R. Sharma, 2018]

$$\kappa_2^B = 0.01689$$
$$\kappa_4^B = 0.00014$$

3-flavor NJL model

- The Lagrangian

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}\left(i\gamma_\mu\partial^\mu - \hat{m}\right)\psi + \hat{\mu}\psi^\dagger\psi + \mathcal{L}_{\mathrm{S}} + \mathcal{L}_{\mathrm{D}},$$

$$\mathcal{L}_{\mathrm{S}} = G_s \sum_{a=0}^{8}\left[\left(\bar{\psi}\lambda_a\psi\right)^2 + \left(\bar{\psi}\, i\gamma_5\lambda_a\psi\right)^2\right],$$

$$\mathcal{L}_{\mathrm{D}} = -G_d\left[\det\bar{\psi}_i(1-\gamma_5)\psi_j + \det\bar{\psi}_i(1+\gamma_5)\psi_j\right].$$

- Under meanfield approximation

$$\mathcal{L}_{\mathrm{MFA}} = \bar{\psi}\left(i\gamma_\mu\partial^\mu + \hat{\mu}\gamma_0 - \hat{M}\right)\psi - 2G_s\sum_i \sigma_i^2 + 4G_d\prod_i \sigma_i$$

- The constituent quark masses

$$M_i = m_i - 4G_s\sigma_i + 2G_d\epsilon_{ijk}\sigma_j\sigma_k$$

3-flavor NJL model

- The free energy

$$\Omega(\sigma_i, T, \mu) = \Omega_{\text{MF}}(\sigma_i) + \Omega_{\text{Vac}}(\sigma_i) + \Omega_{\text{Th}}(\sigma_i, T, \mu),$$

where

$$\Omega_{\text{MF}} = 2G_s \sum_i \sigma_i^2 - 4G_d \prod_i \sigma_i,$$

$$\Omega_{\text{Vac}} = -2N_c \sum_i \int^{\Lambda} \frac{d^3p}{(2\pi)^3} E_i(p),$$

$$\Omega_{\text{Th}} = -2N_c T \sum_i \int \frac{d^3p}{(2\pi)^3} \left[\ln\left(1 + e^{-(E_i(p)-\mu_i)/T}\right) + (\mu_i \to -\mu_i)\right]$$

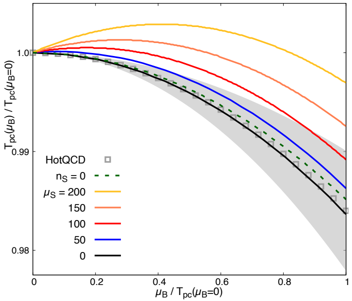
- Minimize the potential for a fixed (T, μ)

$$\frac{\partial\Omega}{\partial\sigma_u} = \frac{\partial\Omega}{\partial\sigma_d} = \frac{\partial\Omega}{\partial\sigma_s} = 0.$$

- For a fixed μ, obtain the inflection point in T (T_{co}) from σ_u vs. T curve.

Phase Diagram

- The phase diagram



- Curvature coefficients are obtained by fitting the ansatz.

$$\frac{T_{co}(\mu_X)}{T_{co}(0)} = 1 - \kappa_2^X \tilde{\mu}_X^2 - \kappa_4^X \tilde{\mu}_X^4$$

- The curvature of the $\mu_S = 0$ line [**MSA**, D. Biswas, A. Jaiswal, and H. Mishra, 2024; A. Bazavov et al. (HotQCD), 2019]

	2f NJL (m=0)	3fNJL	LQCD
κ_2^B $(\mu_S = 0)$	0.01689	0.01627	0.016(6)

Curvature Coefficients

- Different model parameters

	Λ (MeV)	$G_s\Lambda^2$	$G_d\Lambda^5$	m_l(MeV)	m_s(MeV)
Set I	631.4	1.835	9.29	5.5	135.7
Set II	602.3	1.835	12.36	5.5	140.7

- κ's for different lines [**MSA**, D. Biswas, A. Jaiswal, and H. Mishra, 2024; A. Bazavov et al. (HotQCD), 2019; S. Borsanyi et al. 2020]

	$\kappa_2^B\ (\mu_S = 0)$	$\kappa_2^S(\mu_B = 0)$	$\kappa_2^{B,n_S=0}$
NJL, set I	0.0163	0.0134	0.0148
NJL, set II	0.0162	0.0172	0.0143
Lattice QCD	0.016(6)	0.017(5)	0.012(4) 0.0153(18)

Curvature from analytic structure

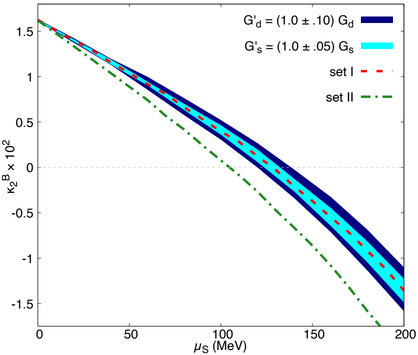
- The gap equation for σ_i

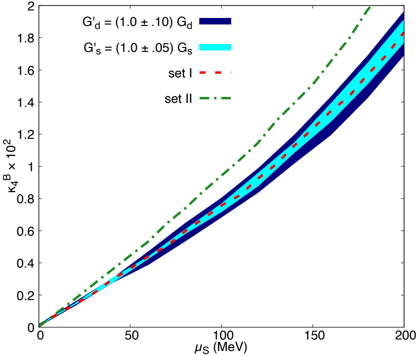
$$\frac{\partial \Omega}{\partial \sigma_i} = \frac{\partial \Omega_{\text{MF}}}{\partial \sigma_i} + \frac{\partial \Omega_{\text{Vac}}}{\partial \sigma_i} + \frac{1}{8\pi^2} \sum_j \int p^2 \, dp \left[\frac{1}{1 + e^{(E_j(p) - \mu_j)/T}} + \frac{1}{1 + e^{(E_j(p) + \mu_j)/T}} \right] \frac{M_j}{E_j} \frac{\partial M_j}{\partial \sigma_i} = 0$$

- Along the phase line, the mean fields (σ_i) are constant up to $\mu_B/T \simeq 2$.
- Similar expression as the 2 flavor chiral case with p replaced by $E(p)$ with constant mass.
- A nonlocal model brings T, μ dependencies to the quark masses, which can bring new effects on curvature coefficients of $T - \mu$ phase diagram.

Constrains on NJL model parameters

- The curvature coefficient κ_2^B as a function of μ_S [**MSA**, D. Biswas, A. Jaiswal, and H. Mishra, 2024]





- Future LQCD data will help us constrain the NJL model (G_d).

Vector Interaction

- The vector interaction term

$$\mathcal{L}_{\text{Vec}} = \begin{cases} -G_V(\bar{q}\gamma^\mu q)^2 \\ -g_V \sum_{a=0}^{8} \left[(\bar{q}\gamma^\mu \lambda_a q)^2 + (\bar{q}i\gamma^\mu\gamma^5\lambda_a q)^2\right] \end{cases}$$

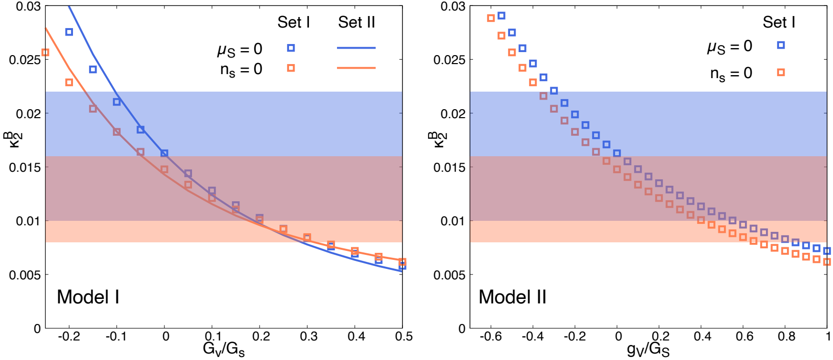
- the chemical potential is modified to

$$\mu_i^* = \begin{cases} \mu_i - 2G_V \sum_j n_j \\ \mu_i - 2g_V n_i \end{cases} .$$

- For degenerate quark masses, $M_u = M_d = M_s$, and $\mu_S = \mu_Q = 0$, $n_u = n_d = n_s$, implying that $g_V = 3G_V$.
- Net zero strange quark, $n_s = 0$, implies $g_V = 2G_V$.

Allowed Strength of Vector Interaction

- We constrain the strength of vector interaction using LQCD results

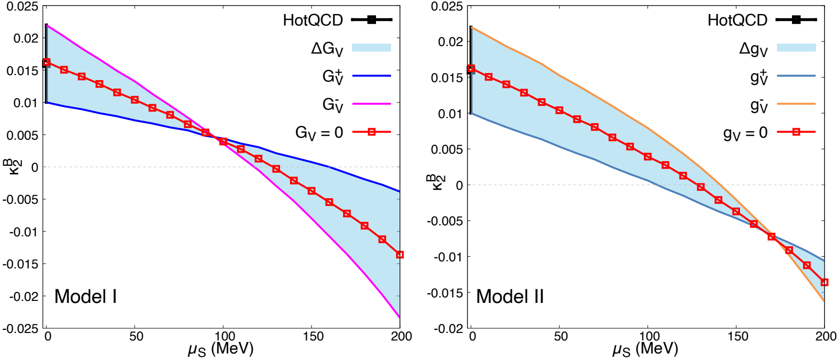


- The allowed range of G_V [**MSA**, D. Biswas, C. A. Islam, 2025]

	κ_2^B $(\mu_S = 0)$	$\kappa_2^{B,n_S=0}$
Lattice QCD	0.016(6)	0.012(4)
G_V	$[-0.233241, 0.424701]$	$[-0.076409, 0.667915]$
g_V	$[-0.592698, 1.105132]$	$[-0.153077, 1.334883]$

Constraints from μ_S Dependency

- Within the allowed range, we explore the effect of μ_S on the curvature coefficient[**MSA**, D. Biswas, C. A. Islam, 2025]

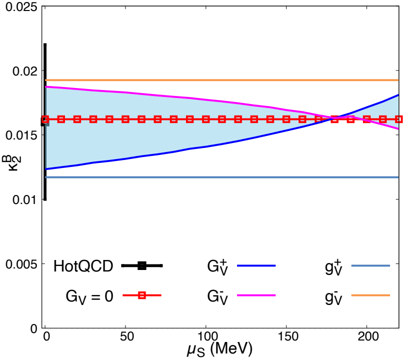


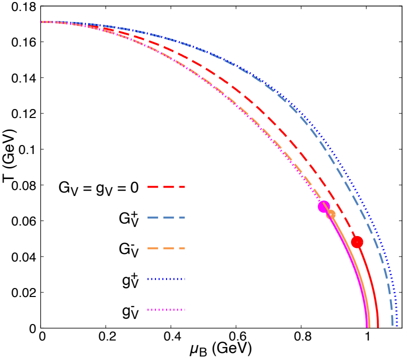
- At certain μ_S, the effect of G_V in getting nullified.
- LQCD results of κ_2^B as a function of μ_S will improve these findings.

Differentiate Different Type of Interaction

- Apart from G_d, flavor-independent vector interaction also introduces flavor mixing between light and strange quark.

- Flavor mixing due vector interaction with $G_d = 0$





- The critical endpoint depends on the strength of vector interaction.

Summary

- The characteristics of the transition line are independent of model parameters when the μ_q's are identical for all quarks.
- We have an excellent agreement of κ_2^B with the available LQCD finding.
- κ_2^B as a function of μ_S and κ_2^S can be used to study the flavor mixing and $U(1)_A$ breaking 't Hooft interaction, G_d.
- Once explored in LQCD, we can constrain the NJL model to a better version of itself.
- Previously unconstrained vector interaction strength can also be constrained once the proposed quantities are explored in LQCD.
- In an effective model scenario, one can distinguish between flavor independent and dependent vector interaction from κ_2^B vs. μ_S data.

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Thank You

Spontaneous Symmetry Breaking

- A symmetry is spontaneously broken when the Lagrangian is symmetric under that symmetry, but the ground state does not.
- Lagrangian of scalar ϕ^4 theory

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

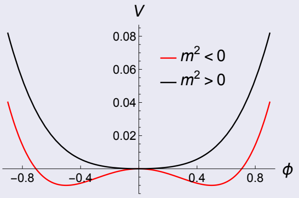
- Symmetric under Z_2, $\phi \to -\phi$.
- The potential

$$V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

- With $m^2 = c(T - T_c)$ and $T < T_c$, the minimum of the potential is at $(m^2 \to -\tilde{m}^2)$

$$\phi_{min} = \pm\sqrt{\frac{6\tilde{m}^2}{\lambda}}.$$

Potential



Spontaneous Symmetry Breaking

- Expanding around the minima

$$\phi = \sqrt{\frac{6\tilde{m}^2}{\lambda}} + \tilde{\phi}$$

- The Lagrangian expanded around the minima is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 + \frac{3\tilde{m}^4}{2\lambda} - \tilde{m}^2\tilde{\phi}^2 - \sqrt{\frac{\lambda}{6}}\tilde{m}\tilde{\phi}^3 - \frac{\lambda}{4!}\tilde{\phi}.$$

- The above Lagrangian is not invariant under $\tilde{\phi} \to -\tilde{\phi}$. Is the symmetry gone?
- Not really! It is symmetric under

$$\tilde{\phi} \to -\tilde{\phi} - 2\sqrt{\frac{6\tilde{m}^2}{\lambda}}.$$

Symmetries (Backup)

- The chiral projection operators are

$$P_R = \frac{1}{2}(1+\gamma_5) \quad \text{and} \quad P_L = \frac{1}{2}(1-\gamma_5)$$

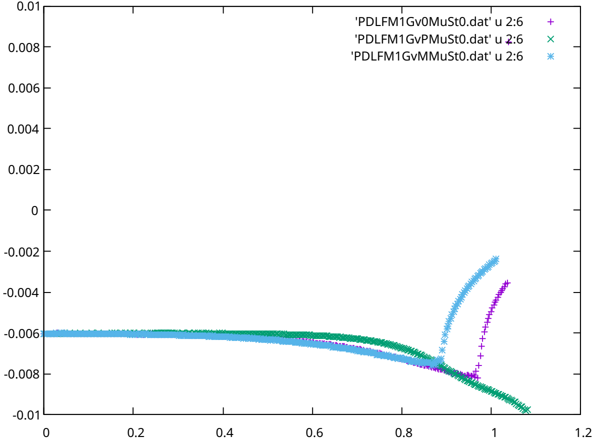
- The quark doublet is given by $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$. With $\psi_R = P_R\psi$ and $\psi_L = P_L\psi$.
- The $SU(2)_L \times SU(2)_R$ symmetry transforms are

$$\psi_L \to e^{i\alpha_L^a \tau^a}\psi_L \quad \text{and} \quad \psi_R \to e^{i\alpha_R^a \tau^a}\psi_R.$$

$\vec{\tau}$'s are the Pauli matrix.
- With little rearrangement, this transformation can be written as

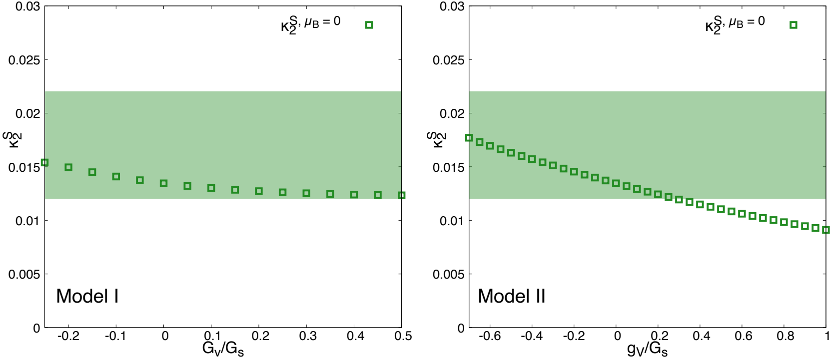
$$\psi \to e^{i\alpha_V^a \tau^a}\psi \quad \text{and} \quad \psi \to e^{i\alpha_A^a \gamma_5 \tau^a}\psi.$$

Meanfield on the Phase Line(Backup)



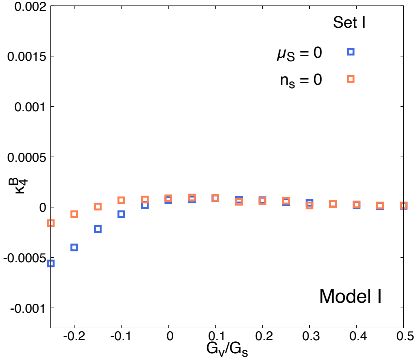
- Remains constant within the range of interest, $\mu_B / T_{co}(0) < 1$.

κ_2^S Vs. G_V/g_V (Backup)



- Very sensitive to the strength of $U(1)_A$ breaking, G_d.
- Mild variation as a function of the strength of vector interactions.
- Does not improve the constraints from κ_2^B.

κ_4^B Vs. G_V/g_V (Backup)



- The LQCD estimations for κ_4^B are 0.001(7) and 0.004(6) for $\mu_S = 0$ and $n_s = 0$, respectively.
- Does not improve the constraints from κ_2^B.

κ_4^B as a function of μ_S with G_V/g_V(Backup)