Relativistic Spin Hydrodynamics in General Frame

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November 11, 2025

Based On : PRC **111** (2025), 034909 and PRC **112** (2025), L021901

Section Outline :

Statement of Problems

Relativistic Kinetic Theory :

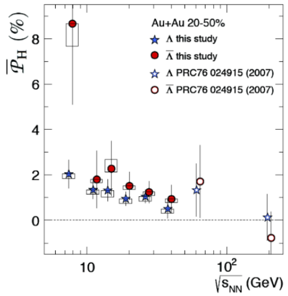
Relativistic Spin Hydrodynamics in General Frame:

Summary and Outlook :

Particle Polarization :

- Spin polarization of hadrons in heavy-ion collisions, predicted in 2004.
  [Z. T. Liang, X. N. Wang, PRL **94**, 102301 (2005), PLB **629**, 20 (2005)]

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions*, Nature 548 62-65, 2017



First evidence of a quantum effect in (relativistic) hydrodynamics

Experimental evidence, [STAR Collaboration, Nature 548, 62 (2017), PRL 123, 132301 (2019), PRL 126, 162301 (2021)]

Theoretical models assuming equilibration of spin d.o.f. explains the data.

Particle Polarization :

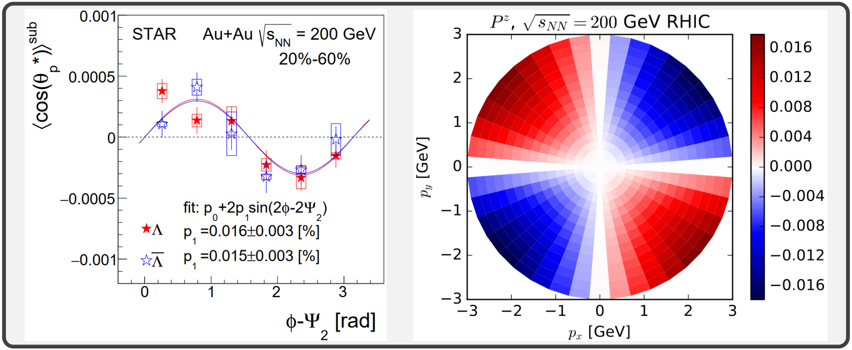


Figure 1: Observation (L) and prediction (R) of longitudinal polarization.

[Left: Phys. Rev. Lett. **123** 132301 (2019); Right: Phys. Rev. Lett. **120** 012302 (2018)]

- Inclusion of shear-induced polarization (SIP) solves the problem with extra constraints.
  [Fu et. al. Phys. Rev. Lett. 127, 142301 (2021); Becattini et. al. Phys. Lett. B 820 136519 (2021)]
- Still the resolution remains ambiguous.
  [Florkowski et. al., Phys. Rev. C 100, 054907 (2019); Phys.Rev.C 105, 064901 (2022)]
- Do dissipative forces play any role and solve the problem?

Spin NRTA Transport :

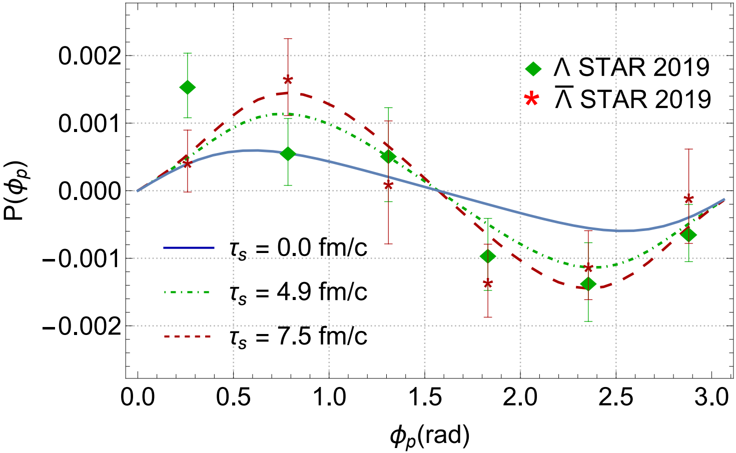


Figure 2: Polarization. $\tau_s = 7.5$ fm for $\bar{\Lambda}$ and, $\tau_s = 4.9$ fm for Λ. [S. Banerjee *et. al.*, accepted in PRC]

- τ_s is in agreement with [Hidaka et. al., PRC **109** (2024), 054909, Wagnar et. al., PRR **6**, (2024) 043103].
- Incorporation of dissipative effects is necessary. [Sapna et. al., arXiv:2503.22552] (Also see Talk by **Sushant Kumar Singh**)

Relativistic Dissipative Spin Hydrodynamics :

- We first note that spin-polarization originates from the rotation of fluid.

- Hence, we will have to deal with three conserved currents :

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad \partial_\lambda J^{\lambda,\mu\nu} = 0$$

where, $J = L + S$. Also, $L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$.

- For symmetric $T^{\mu\nu}$ we have, $\boxed{\partial_\lambda S^{\lambda,\mu\nu} = 0}$

$$N^\mu = N^\mu_{\text{eq}} + \delta N^\mu, \qquad T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \delta T^{\mu\nu}, \qquad S^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}_{\text{eq}} + \delta S^{\lambda,\mu\nu}$$

- The dissipative parts require microscopic description $\rightarrow$ Kinetic Theory.

Relativistic Dissipative Spin Hydrodynamics :

- Hydrodynamics is derived through an order-by-order gradient expansion:

$$X^{\mu_1\cdots\mu_\ell} = X^{\mu_1\cdots\mu_\ell}_{(0)} + X^{\mu_1\cdots\mu_\ell}_{(1)} + \cdots + X^{\mu_1\cdots\mu_\ell}_{(n)}$$

$$\begin{aligned} n = 0 &\longrightarrow \quad \text{Perfect fluid dynamics.} \\ n = 1 &\longrightarrow \quad \left.\begin{aligned} &\text{Navier} - \text{Stokes.} \\ &\text{MIS} - \text{type theories.} \end{aligned}\right\}\text{Dissipative} \\ n = 2 &\longrightarrow \end{aligned}$$

- Navier-Stokes theories lead to acausal theories.

  [W. A. Hiscock et. al., Annals Phys. **151** (1983) 466-496, PRD **31** (1985) 725-733, **35** (1987) 3723-3732]

- MIS theories introduce extra degrees of freedom.

- Casual 1st-order theories require generalized frame definition - BDNK Theory.

  [F. Bemfica et. al., PRD **98** (2018), 104064 PRL **122** (2019), 221602,
  P. Kovtun JHEP **10** (2019) 034, PRD **106** (2022), 066023]

Section Outline :

Kinetic Theory

- The central object within relativistic kinetic theory is the phase-space distribution function, $f(x,p,s)$.

- Different moments of $f(x,p)$ correspond to different hydrodynamical variables,

  $N^{\mu}=\int dPdS\, p^{\mu}\,(f-\bar{f}), \quad T^{\mu\nu}=\int dPdS\, p^{\mu}p^{\nu}\,(f+\bar{f}), \quad S^{\lambda,\mu\nu}=\int dPdS\, p^{\lambda}s^{\mu\nu}\,(f+\bar{f})$

  where, $dP \equiv \mathrm{d}^3\mathbf{p}/(2\pi)^3 p^0$, $dS \equiv \frac{m}{\pi\mathfrak{s}} d^4s\,\delta(s\cdot s+\mathfrak{s}^2)\delta(p\cdot s)$, and p^0 $(=\sqrt{\mathbf{p}^2+m^2})$ is the particle energy.

- Dissipative quantities are defined using non-equilibrium correction to $f(x,p)$.

  $$f(x,p,s) = f_{\rm eq}(x,p,s) + \delta f(x,p,s) = f_{\rm eq}\,(1+\phi_s)$$

  [Landau and Lifshitz, Fluid Dynamics; P. Romatschke, IJMPE 19 (2010) 1-53]

- Equilibrium quantities are defined using equilibrium distribution function.

  $$f_{\rm eq} = (e^{g}+a)^{-1}$$

  where, $a=0,\pm 1$ and $g \equiv \sum_n \alpha_n \phi_n$.

  [e.g.- $g = (\beta\cdot p) - \xi + \frac{(s:\omega)}{2}, \beta^{\mu} \equiv u^{\mu}/T, \xi \equiv \mu/T$]

Boltzmann Equation and The Collision Kernel

- Expression of $\delta f(x,p)$ is required. Obtained from the Boltzmann equation,

$$p^\mu \partial_\mu f + F^\mu \partial_\mu^{(p)} f = C[f]$$

where, $C[f]$ is the collision kernel that controls the interaction process.
p^μ is the particle momentum and F^μ is the background force.

[de Groot, van Leeuwen and, van Weert, Relativistic kinetic Theory]

[C. Cercignani, Mathematical Methods in Kinetic Theory K. Huang, Statistical Mechanics]

- For $2 \leftrightarrow 2$ collisions:

$$C[f] = \int dP\, dP'\, dK'\, \underbrace{\mathcal{W}_{\mathbf{kk'}\leftrightarrow\mathbf{pp'}}} \times \left(f_\mathbf{p} f_\mathbf{p'} - f_\mathbf{k} f_\mathbf{k'}\right)$$

Transition Amplitude:

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Transition Amplitude:

- Linearized Collision Operator :

$$C[f] \to \hat{L}\phi_{\mathbf{k}} = \int dP\, dP'\, dK'\, \mathcal{W}_{\mathbf{kk'}\leftrightarrow\mathbf{pp'}} f_{0\mathbf{k}} f_{0\mathbf{k'}} \times \left(\phi_{\mathbf{p}} + \phi_{\mathbf{p'}} - \phi_{\mathbf{k}} - \phi_{\mathbf{k'}}\right)$$

Collisional Invariants:

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

The Conservation Laws

- Collisional invariants remain conserved during collisions.

- Each collisional invariant correspond to a conservation law.

- For a non-rotating, unpolarizable fluid :
  - $\phi \equiv 1 \longrightarrow$ Number Conservation.
  - $\phi \equiv E_{\mathbf{k}} \longrightarrow$ Energy Conservation.
  - $\phi \equiv \vec{k} \, (\sim k^{\langle\mu\rangle}) \longrightarrow$ Linear Momentum Conservation.

- Thus, a collision kernel should satisfy:

$$\hat{L}\, 1 = 0, \qquad \hat{L}\, E_{\mathbf{k}} = 0, \qquad \hat{L}\, k^{\langle\mu\rangle} = 0.$$

- The linearized collision kernel satisfies the property:

$$\int dK \, \psi_{\mathbf{k}} \hat{L} \phi_{\mathbf{k}} = \int dK \, \phi_{\mathbf{k}} \hat{L} \psi_{\mathbf{k}}$$

$$\Longrightarrow \int dK \, \hat{L}\phi_{\mathbf{k}} = 0, \qquad \int dK \, k^{\mu} \hat{L}\phi_{\mathbf{k}} = \int dK \left(u^{\mu} E_{\mathbf{k}} + k^{\langle\mu\rangle} \right) \hat{L}\phi_{\mathbf{k}} = 0.$$

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

New Collision Kernels

- Most widely used collision kernel is the Relaxation Time Approximation (RTA):

$$k^\mu \partial_\mu f = -\frac{(u\cdot k)}{\tau_{\rm R}}\,(f - f_{0\mathbf{k}}) = -\frac{E_{\mathbf{k}}}{\tau_{\rm R}}\,\phi_{\mathbf{k}} f_{0\mathbf{k}}$$

- We will work with two types linearized collision kernels:

1. Extended Relaxation Time Approximation (ERTA):

$$\boxed{\hat{L}_{\rm ERTA}\phi_{\mathbf{k}} = -\frac{E_{\mathbf{k}}}{\tau_{\rm R}}\,(\phi_{\mathbf{k}} - \phi^*_{\mathbf{k}})\,f_{0\mathbf{k}}}$$

where,

$$\phi^*_{1,\mathbf{k}} = -\frac{(k\cdot\delta u)}{T} + \frac{(E_{\mathbf{k}}-\mu)\,\delta T}{T^2} + \frac{\delta\mu}{T}$$
$$\delta u_\mu = u^*_\mu - u_\mu, \qquad \delta T = T^* - T, \qquad \delta\mu = \mu^* - \mu.$$

[D. Dash *et. al.*, PLB 831 (2022) 137202]

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[D. Dash *et. al.*, PLB 831 (2022) 137202]

2. Novel Relaxation Time Approximation (NRTA):

$$\boxed{\hat{L}_{\rm NRTA} \sim \left(\mathbb{1} - \sum_{n=1}^{5} |\lambda_n\rangle \langle \lambda_n| \right)}$$

where, $|\lambda_n\rangle$ are degenerate, orthogonal eigenvectors of $\hat{L}_{\rm NRTA}$.

[G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

Solving The Boltzmann Equation

- Boltzmann equation is an inhomogeneous differential equation.

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- Boltzmann equation is an inhomogeneous differential equation.
- Thus, the solution is:

$$\phi = \phi_{\rm h} + \phi_{\rm ih}$$

- Homogeneous part for spin-polarizable particles :

$$\phi_{\rm h} = a + b_\mu k^\mu + c_{\mu\nu} s^{\mu\nu}$$

- Coefficients, $a, b_\mu, c_{\mu\nu}$ can be determined from frame and matching conditions.

- Subtracting the homogeneous part of the solution, gives freedom to choose the frame and matching conditions.

  [S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

- Two popular approaches are considered :
  1. Chapman-Enskog-like iterative solution: Used for ERTA.
     [D. Dash *et. al.*, PLB 831 (2022) 137202]
  2. Moment method: Used for NRTA.
     [G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

Section Outline :

Field Redefinition - ERTA

- The thermodynamic variables (starred) are determined via field re-definitions :

$$\langle q_1\,\phi_s\rangle_0 + \langle \bar{q}_1\,\bar{\phi}_s\rangle_{\bar{0}} = 0, \qquad \langle q_2\,\phi_s\rangle_0 + \langle \bar{q}_2\,\bar{\phi}_s\rangle_{\bar{0}} = 0$$
$$\left\langle q_3\,k^{\langle\mu\rangle}\phi_s\right\rangle_0 + \left\langle \bar{q}_3\,k^{\langle\mu\rangle}\bar{\phi}_s\right\rangle_{\bar{0}} = 0, \qquad \langle q_4\,s^{\mu\nu}\phi_s\rangle_0 + \langle \bar{q}_4\,s^{\mu\nu}\bar{\phi}_s\rangle_{\bar{0}} = 0$$

  where we use the notations:

$$\langle(\cdots)\rangle_0 = \int dK\,dS\,(\cdots)\,f_{0\mathbf{k}}, \qquad \langle(\cdots)\rangle_{\bar{0}} = \int dK\,dS\,(\cdots)\,\bar{f}_{0\mathbf{k}}$$

- Then we find using the field-redefinitions (up to $\mathcal{O}(\partial)$):

$$\delta u_\mu = \beta\mathcal{C}_1\left(\nabla_\mu\xi\right), \qquad \delta T = \mathcal{C}_2\,\theta, \qquad \delta\mu = \mathcal{C}_3\,\theta,$$
$$\delta\omega^{\mu\nu} = \mathcal{D}_{\Pi}^{\mu\nu}\,\theta + \mathcal{D}_{n}^{\mu\nu\gamma}\left(\nabla_\gamma\xi\right) + \mathcal{D}_{\pi}^{\mu\nu\alpha\beta}\,\sigma_{\alpha\beta} + \mathcal{D}_{\Sigma}^{\mu\nu\gamma\alpha\beta}\left(\nabla_\gamma\omega_{\alpha\beta}\right).$$

- Even with arbitrary frame and matching conditions, the first-order theory is still acausal.

The Linearized collision kernel - NRTA

- We work with $\mu = 0$, as NRTA is not built for pair production and annihilation.

- We have 10 collisional invariants and hence : $\hat{L}\phi_s \equiv -\mathbb{1} + \sum_{n=1}^{10} |\lambda_n\rangle \langle\lambda_n|\phi_s\rangle$

- The collision kernel is given by:

$$\begin{aligned}
\hat{L}\phi_s = &-\left(\frac{E_{\mathbf{p}}}{\tau_{\mathrm{R}}}\right) f_{0s}\Bigg\{\phi_s - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\phi_s\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{p}} - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\mu\rangle}\phi_s\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\alpha\rangle} p_{\langle\alpha\rangle}\right\rangle_0} p_{\langle\mu\rangle} \\
&- \left[\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\widetilde{s}_{\mu}\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}_{\mu}\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0}\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\gamma\rangle}\widetilde{s}_{\mu}\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\alpha\rangle} p_{\langle\alpha\rangle}\right\rangle_0}\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p_{\langle\gamma\rangle}\phi_s\right\rangle_0\right] \\
&\times\left[\widetilde{s}^{\mu} - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\mu}\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{p}} - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\rho\rangle}\widetilde{s}^{\mu}\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\beta\rangle} p_{\langle\beta\rangle}\right\rangle_0} p_{\langle\rho\rangle}\right]\frac{1}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\left(\widetilde{s}\cdot\widetilde{s}\right)\right\rangle_0} \\
&- \left[\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\widetilde{s}_{\mu\nu}\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}_{\mu\nu}\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0}\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\gamma\rangle}\widetilde{s}_{\mu\nu}\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\alpha\rangle} p_{\langle\alpha\rangle}\right\rangle_0}\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p_{\langle\gamma\rangle}\phi_s\right\rangle_0\right] \\
&\times\left[\widetilde{s}^{\mu\nu} - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\mu\nu}\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{p}} - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\rho\rangle}\widetilde{s}^{\mu\nu}\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\beta\rangle} p_{\langle\beta\rangle}\right\rangle_0} p_{\langle\rho\rangle}\right]\frac{1}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\left(\widetilde{s}:\widetilde{s}\right)\right\rangle_0}\Bigg\},
\end{aligned}$$

where, $\widetilde{s}^{\mu} = \Delta^{\mu\alpha} u^{\beta} s_{\alpha\beta}$, and $\widetilde{s}^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta} s_{\alpha\beta}$.

- This leads to:

$$\hat{L}E_{\mathbf{k}} = 0, \qquad \hat{L}k^{\langle\mu\rangle} = 0, \qquad \hat{L}s^{\mu\nu} = 0.$$

[S. B., PRC **112** (2025), L021901]

Constitutive Equations & Dissipative Currents

- Under a general field re-definition the constitutive relations are given by :

$$N^{\mu} = (n_0 + \delta n)\, u^{\mu} + n^{\mu},$$
$$T^{\mu\nu} = (\mathcal{E}_0 + \delta\mathcal{E})\, u^{\mu}u^{\nu} - (\mathcal{P} + \delta\mathcal{P})\,\Delta^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu},$$
$$S^{\lambda,\mu\nu} = S_0^{\lambda,\mu\nu} + \delta S^{\lambda,\mu\nu}.$$

- The entropy production is given by,

$$\partial_\mu \mathcal{H}_\mu = -\beta \Pi \theta - \mathcal{Q}^{\mu}\left(\nabla_\mu \xi\right) + \beta \pi^{\mu\nu}\sigma_{\mu\nu} - \mathcal{S}^{\lambda\mu\nu}\left(\nabla_\lambda \omega_{\mu\nu}\right)$$

- The frame-invariant transport coefficients are:

$$\Pi = -\zeta\,\theta = \delta\mathcal{P} - \left(\frac{\chi_b}{\beta}\right)\delta\mathcal{E} + \left(\frac{\chi_a}{\beta}\right)\delta n,$$
$$\mathcal{Q}^{\mu} = \kappa^{\mu\nu}\left(\nabla_\nu \xi\right) = n^{\mu} - \left(\frac{n_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) h^{\mu},$$
$$\pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu},$$
$$\mathcal{S}^{\lambda\mu\nu} = \beta_{\Sigma}^{\lambda\mu\nu\gamma\alpha\beta}\left(\nabla_\gamma \omega_{\alpha\beta}\right) = \frac{1}{2}\left(u^{\rho} D_{\Sigma}^{\alpha\beta\lambda\mu\nu}\delta S_{\rho,\alpha\beta} - \delta S^{\lambda,\mu\nu}\right)$$

[S. B., PRC **111** (2025), 034909]

Section Outline :

Summary and Outlook :

- **Summary :**
  1. ERTA allows us to have a general choice of frame and matching conditions.
  2. ERTA does not lead to first-order causal theory of spin hydrodynamics.
  3. NRTA gives the option of constructing first-order causal spin-hydrodynamics.
  4. NRTA cannot describe a system with pair production and annihilation.

- **Outlook :**
  1. Construction of spin-BDNK theory is required.
  2. Perform a linear mode analysis to verify the causality and stability of the theory.

Thank you.

Back Up :

- The frame-invariant transport coefficients are:

$$f_i \equiv \pi_i - \left(\frac{\partial \mathcal{P}_0}{\partial \mathcal{E}_0}\right)_{n_0} \varepsilon_i + \left(\frac{\partial \mathcal{P}_0}{\partial n_0}\right)_{\mathcal{E}_0} \nu_i,$$
$$l_i \equiv \gamma_i - \left(\frac{n_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) \theta_i.$$

- Out of the 16 parameters, only three one-derivative transport coefficients $\zeta(f_1, f_2, f_3),\ \eta,\ \kappa(l_1, l_2)$ are independent.

Inclusion of Spin

- Phase-space is extended to include spin degrees of freedom :

$$f_{\mathbf{k}}(x,k) \longrightarrow f_s(x,k,s)$$
$$f_{0\mathbf{k}} \longrightarrow f_{0,s} = f_{0\mathbf{k}} \exp(s:\omega) \approx f_{0\mathbf{k}} \left[1 + \frac{1}{2}(s:\omega)\right] + \mathcal{O}\left(\omega^2\right)$$

- Homogeneous part for spin-polarizable particles :

$$\phi_{\mathrm{h}} = a + b_\mu k^\mu + c_{\mu\nu} s^{\mu\nu}$$

- The solutions are modified as :

1. Chapman-Enskog-like iterative solution (ERTA):

$$\phi_{n,s} f_{0,s} = \phi^*_{n,s}\, f_{0,s} - \frac{\tau_{\mathrm{R}}}{(u\cdot p)}\,(p\cdot\partial)\, f_{(n-1),s},$$

[S. B., PRC **111** (2025), 034909]

2. Moment method (NRTA):

$$\phi_s = \sum_{n,\ell=0}^{\infty}\left(\Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} + s_{\mu\nu}\Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle}\right) k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} P_n^{(\ell)}(\beta E_{\mathbf{k}})$$

[S. B., PRC **112** (2025), L021901]

Field Redefinition - NRTA

- We work with $\mu = 0$, as NRTA is not built for pair production and annihilation.
- Now we use a new set of notations :

$$\langle(\cdots)\rangle_{\mathrm{o}} = \int dK\, dS\, (\cdots)\, f_{\mathrm{o}s}, \qquad \langle(\cdots)\rangle_{\mathrm{o}\mathbf{k}} = \int dK\, (\cdots)\, f_{\mathrm{o}\mathbf{k}}$$

- The thermodynamic variables (starred) are determined via field redefinitions:

$$\int dK dS\, q_1 \phi_s\, f_{\mathrm{o}\mathbf{k}} = 0, \qquad \int dK dS\, q_2 k^{\langle\mu\rangle} \phi_s\, f_{\mathrm{o}\mathbf{k}} = 0, \qquad \int dK dS\, q_3 s^{\mu\nu} \phi_s\, f_{\mathrm{o}\mathbf{k}} = 0$$

- The solution takes the form:

$$\phi_s = \phi_{\mathbf{p}} + s : \psi_{\mathbf{p}}.$$

with

$$\phi_{\mathbf{p}} = \sum_{n\in\mathbb{S}_0^{(\ell)}} \sum_{\ell=0}^{\infty} \Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} p_{\langle\mu_1} \cdots p_{\mu_\ell\rangle} P_n^{(0,\ell)}, \qquad \text{and,} \qquad \psi_{\mathbf{p}}^{\mu\nu} = \sum_{n\in\mathbb{S}_1^{(\ell)}} \sum_{\ell=0}^{\infty} \Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle} p_{\langle\mu_1} \cdots p_{\mu_\ell\rangle} P_n^{(1,\ell)},$$

$$\frac{\ell!}{(2\ell+1)!!} \int dP\, (E_{\mathbf{p}}/\tau_{\mathrm{R}})\, (p\cdot\Delta\cdot p)^{\ell}\, P_m^{(j,\ell)} P_n^{(j,\ell)} f_{\mathrm{o}\mathbf{p}} = A_n^{(j,\ell)} \delta_{mn}.$$

[S. B., 2503.08428]

Solution

- The homogeneous parts of the solution are :

$$\Phi_0 = -\sum_{n \in \mathbb{S}_0^{(0)} - \{0\}} \frac{\left\langle q_1 \, P_n^{(0,0)} \right\rangle_{0\mathbf{p}}}{\left\langle q_1 \, P_0^{(0,0)} \right\rangle_{0\mathbf{p}}} \Phi_n, \qquad \Phi_0^{\langle \mu \rangle} = -\frac{\left\langle q_2 \, (p \cdot \Delta \cdot p) \, P_1^{(0,1)} \right\rangle_{0\mathbf{p}}}{\left\langle q_2 \, (p \cdot \Delta \cdot p) \, P_0^{(0,1)} \right\rangle_{0\mathbf{p}}} \Phi_1^{\langle \mu \rangle},$$

$$\Psi_0^{\mu\nu} = -\left[\frac{\left\langle q_3 P_1^{(1,0)} \right\rangle_{0\mathbf{p}}}{\left\langle q_3 P_0^{(1,0)} \right\rangle_{0\mathbf{p}}} \right] \Psi_1^{\mu\nu}.$$

- Using these, one can obtain the expressions of all $\Phi_n^{\langle \mu_1 \cdots \mu_\ell \rangle}$ and $\Psi_n^{\mu\nu, \langle \mu_1 \cdots \mu_\ell \rangle}$.
  [S. B., 2503.08428]

Inclusion of Dissipation :

- Full collision kernel for Novel RTA with spin :
  [S. B. arXiv:2503.08428]

$$
\begin{aligned}
\hat{L}\phi_s = &-\left(\frac{E_{\mathbf{p}}}{\tau_{\mathrm{R}}}\right) f_{0s}\Bigg\{\phi_s - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\phi_s\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{p}} - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\mu\rangle}\phi_s\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\alpha\rangle} p_{\langle\alpha\rangle}\right\rangle_0} p_{\langle\mu\rangle} \\
&- \left[\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\widetilde{s}_\mu\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}_\mu\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0}\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\gamma\rangle}\widetilde{s}_\mu\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\alpha\rangle} p_{\langle\alpha\rangle}\right\rangle_0}\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p_{\langle\gamma\rangle}\phi_s\right\rangle_0\right] \\
&\quad\times\left[\widetilde{s}^\mu - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^\mu\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{p}} - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\rho\rangle}\widetilde{s}^\mu\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\beta\rangle} p_{\langle\beta\rangle}\right\rangle_0} p_{\langle\rho\rangle}\right]\frac{1}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\left(\widetilde{s}\cdot\widetilde{s}\right)\right\rangle_0} \\
&- \left[\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\widetilde{s}_{\mu\nu}\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}_{\mu\nu}\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0}\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\phi_s\right\rangle_0 - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\gamma\rangle}\widetilde{s}_{\mu\nu}\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\alpha\rangle} p_{\langle\alpha\rangle}\right\rangle_0}\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p_{\langle\gamma\rangle}\phi_s\right\rangle_0\right] \\
&\quad\times\left[\widetilde{s}^{\mu\nu} - \frac{\left\langle\left(E_{\mathbf{p}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\mu\nu}\right\rangle_0}{\left\langle\left(E_{\mathbf{p}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{p}} - \frac{\left\langle\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\rho\rangle}\widetilde{s}^{\mu\nu}\right\rangle_0}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right) p^{\langle\beta\rangle} p_{\langle\beta\rangle}\right\rangle_0} p_{\langle\rho\rangle}\right]\frac{1}{\left\langle(1/3)\left(E_{\mathbf{p}}/\tau_{\mathrm{R}}\right)\left(\widetilde{s}:\widetilde{s}\right)\right\rangle_0}\Bigg\},
\end{aligned}
$$

Inclusion of Dissipation :

- Decomposition of conserved current :

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu - P\,\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} = \langle p^\mu p^\nu \rangle\,,$$

$$S^{\lambda,\mu\nu} = u^\lambda \mathcal{S}^{\mu\nu} + \left(u^\mu \Delta^{\nu\lambda} - u^\nu \Delta^{\mu\lambda}\right)\Sigma + u^\mu \Sigma^{\langle\lambda\nu\rangle}_{(s)} - u^\nu \Sigma^{\langle\lambda\mu\rangle}_{(s)} + u^\mu \Sigma^{\lambda\nu}_{(a)} - u^\nu \Sigma^{\lambda\mu}_{(a)} + \Sigma^{\lambda,\mu\nu} = \left\langle p^\lambda s^{\mu\nu}\right\rangle,$$

- Transport Coefficients are given by :

$$\delta\varepsilon = \sum_{n\in \mathbb{S}_0^{(0)}} \Phi_n \left\langle E_{\mathbf{p}}^2 P_n^{(0,0)} \right\rangle_{0\mathbf{p}}\,, \qquad \delta P = -\sum_{n\in \mathbb{S}_0^{(0)}} \Phi_n \left\langle (1/3)\,(p\cdot\Delta\cdot p)\, P_n^{(0,0)} \right\rangle_{0\mathbf{p}}\,,$$

$$q^\mu = \sum_{n\in \mathbb{S}_0^{(1)}} \Phi_n^{\langle\mu_1\rangle} \left\langle (1/3)\, E_{\mathbf{p}}\,(p\cdot\Delta\cdot p)\, P_n^{(0,1)} \right\rangle_{0\mathbf{p}}\,, \qquad \pi^{\mu\nu} = \Phi_0^{\langle\mu\nu\rangle} \left\langle (2/15)\,(p\cdot\Delta\cdot p)^2 \right\rangle_{0\mathbf{p}}\,,$$

$$\delta\mathcal{S}^{\mu\nu} = \sum_{n\in \mathbb{S}_1^{(0)}} \Psi_n^{\mu\nu} \left\langle E_{\mathbf{p}} P_n^{(1,0)} \right\rangle_{0\mathbf{p}}\,, \qquad \delta\Sigma^{\langle\mu\nu\rangle}_{(s)} = u_\alpha \Delta^{\mu\nu}_{\rho\gamma} \Psi_0^{\alpha\rho,\langle\gamma\rangle} \left\langle (1/3)\,(p\cdot\Delta\cdot p)\right\rangle_{0\mathbf{p}}\,,$$

$$\delta\Sigma^{\mu\nu}_{(a)} = \sum_{n\in \mathbb{S}_1^{(0)}} u_\alpha \Psi_n^{\alpha[\mu} u^{\nu]} \left\langle E_{\mathbf{p}} P_n^{(1,0)} \right\rangle_{0\mathbf{p}} + u_\alpha \Psi_0^{\alpha[\mu,\langle\nu\rangle]} \left\langle (1/3)\,(p\cdot\Delta\cdot p)\right\rangle_{0\mathbf{p}}\,, \qquad \delta\Sigma^{\lambda,\mu\nu} = \Psi_0^{\langle\mu\rangle\langle\nu\rangle,\langle\lambda\rangle} \left\langle (p\cdot\Delta\cdot p)\right\rangle_{0\mathbf{p}}\,,$$

[S. B. arXiv:2503.08428]