The causality-stability paradox of relativistic hydrodynamics

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Study of hydrodynamics

Motivation behind ⋯

- Hydrodynamics : A long-wavelength effective theory of fluids that describes evolution of the system in terms of it's state variables and their derivatives.
- Riddle : There can be many possible hydrodynamic theories; how can we filter out unphysical ones?

Key features for theory to be physically acceptable ⋯

- Causality: Signal propagation should not exit the light cone; no superluminal velocities.
- Stability: Perturbations around global equilibrium must decay down with time.
- Pathology free: Both stable as well as causal.

Established theories ⋯

- Muller-Israel-Stewart (MIS): Higher order stable-causal theory. Ref: Israel, Annals Phys. 100 (1976), 310-331, Israel and Stewart, Annals Phys. 118, 341-372, Muller, Z. Phys. 198 (1967), 329-344.

Recent attempt and current field of study ⋯

- BDNK: First order stable-causal theory. Ref: Bemfica, Disconzi and Noronha, PRD 98, no.10, 104064 (2018), PRD 100, no.10, 104020 (2019), P. Kovtun, JHEP 10, 034 (2019), JHEP 06, 067 (2020).

Causality and stability - a choice to be endured

Desirable features preferred by a hydrodynamic theory

1. They are defined only in terms of the dynamical variables, such as temperature (T), fluid velocity (u^μ), and chemical potential (μ), that are related to some conserved quantities $\Longrightarrow$ characterizing the equilibrium.
2. Even out of equilibrium, the fluid fields are well defined that can attach some physical meaning to it $\Longrightarrow$ defined in terms of some field theory operators.
3. The final differential equations for the fluid variables must contain a finite number of derivatives $\Longrightarrow$ the theory must be truncated.

The features (i) and (ii) are needed to build an unambiguous connection between the fluid variables and the experimentally measurable quantities.

The third feature (iii) is a must to have computationally tractable evolution equations.

Is the holy trinity ever compatible with causality ?

Causality and stability - a choice to be endured

In reality, the journey of relativistic dissipative hydro is far from this desirable holy trinity

1. Theory defined in terms of fundamental fluid variables (T, u^μ, μ) + Landau or Eckart frame (fluid variables are defined in a physically measurable way) + truncated at finite order $\Longrightarrow$ causality violation.

   $\Downarrow$

   Relativistic first-order (Navier-Stokes) theory : Superluminal signal propagation and thermodynamic instability.

2. Causal theory + second or higher (but finite) order derivative corrections + Landau or Eckart frame (fluid variables have first principle definition)

   $\Downarrow$

   Muller-Israel-Stewart theory : New degrees of freedom (with no equilibrium counterparts) are needed.

3. Causal theory + defined only in terms of fundamental fluid variables (T, u^μ, μ) (no extra 'non-fluid' variables) + truncated at finite order of derivative corrections

   $\Downarrow$

   First order Bemfica-Disconzi-Noronha-Kovtun (BDNK) theory : definition of the fluid variables away from equilibrium are not fixed (theory is pathology-free only in frames other than Landau or Eckart).

4. Causal theory + defined only in terms of fundamental fluid variables (T, u^μ, μ) + Landau frame $\Longrightarrow$ infinite number of derivatives.

   $\Downarrow$

   Practical limitations for simulation purpose.

Causality and stability - a choice to be endured

The current work indicates :

- There is a tension between the three desirable features of hydrodynamics.
- To maintain causality and stability we have to give up at least one of them.

Lets test ! How different hydro formalisms could connect !

BDNK stress tensor : $T^{\mu\nu} = (\varepsilon + \mathcal{A})\left[u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3}\right] + [u^\mu Q^\nu + u^\nu Q^\mu] - 2\eta\sigma^{\mu\nu}$

$$\mathcal{A} = \chi\left[3\frac{DT}{T} + \nabla_\mu u^\mu\right], \quad Q^\mu = \theta\left[\frac{\nabla^\mu T}{T} + Du^\mu\right]$$

Ref : Kovtun, JHEP 10 (2019), 034 , Bemfica et al., PRD 100 (2019), 104020

Fluid frame transformation of BDNK theory : $u^\mu = \hat{u}^\mu + \delta u^\mu, \qquad T = \hat{T} + \delta T$

$$\delta u^\mu = \delta u_1^\mu + \delta u_2^\mu + \cdots = \textstyle\sum_{n=1}^\infty \delta u_n^\mu, \qquad \delta T = \delta T_1 + \delta T_2 + \cdots = \textstyle\sum_{n=1}^\infty \delta T_n$$

Estimating frame transformation order by order once the Landau gauge condition is imposed:

$$\delta T_1 = -\tilde{\chi}\left[\frac{\hat{D}\hat{T}}{\hat{T}} + \frac{1}{3}\hat{\nabla}_\mu \hat{u}^\mu\right], \qquad \delta u_1^\mu = -\tilde{\theta}\left[\frac{\hat{\nabla}^\mu \hat{T}}{\hat{T}} + \hat{D}\hat{u}^\mu\right],$$

$$\delta T_n = -\tilde{\chi}\left[\frac{1}{\hat{T}}\hat{D}\delta T_{n-1} + \frac{1}{3}\hat{\nabla}_\mu \delta u_{n-1}^\mu\right] \quad , \quad \delta u_n^\mu = -\tilde{\theta}\left[\frac{1}{\hat{T}}\hat{\nabla}^\mu \delta T_{n-1} + \hat{D}\delta u_{n-1}^\mu\right] \quad \text{for } n \geq 2$$

Causality and stability - a choice to be endured

BDNK theory in Landau frame : temperature and velocity have first principle definition

$$T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^{\mu}\hat{u}^{\nu} + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu}$$

The price is paid : (i) either by an infinite number of derivatives

$\hat{\pi}^{\mu\nu} = -2\eta\left[\hat{\sigma}^{\mu\nu} + \sum_{n=1}^{\infty}\partial^{\langle\mu}\delta u_n^{\nu\rangle}\right]$: summed over temporal derivatives under the all order frame transformation

$$\hat{\pi}^{\mu\nu} = -2\eta\Bigg[\frac{\hat{\nabla}^{\langle\mu}\hat{u}^{\nu\rangle}}{(1+\tilde{\theta}\hat{D})} + \frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})}\frac{\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{T}}{(1+\tilde{\chi}\hat{D})} + \frac{(-\tilde{\theta})}{(1+\tilde{\theta}\hat{D})^2}\frac{(-\frac{1}{3}\tilde{\chi})}{\left(1+\tilde{\chi}\hat{D}\right)}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}\cdot\hat{u}$$
$$+ \frac{(-\tilde{\theta})^2}{(1+\tilde{\theta}\hat{D})^2}\frac{\left(-\frac{1}{3}\tilde{\chi}\right)}{(1+\tilde{\chi}\hat{D})^2}\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}^2\hat{T} + \frac{(-\tilde{\theta})^2}{(1+\tilde{\theta}\hat{D})^3}\frac{\left(-\frac{1}{3}\tilde{\chi}\right)^2}{(1+\tilde{\chi}\hat{D})^2}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\nu\rangle}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots\Bigg]$$

- In each increasing spatial gradient, the temporal gradient resulting from the infinite sum also increases in the denominator, such that they exactly balance each other $\Longrightarrow$ a necessary condition of causality.
- $\pi^{\mu\nu}$ has time derivatives in the denominator $\Longrightarrow$ nonlocality in time.
- Such nonlocalities can be recast into a local set of equations by introducing new 'non-fluid' variables just like the MIS theory.

Causality and stability - a choice to be endured

BDNK theory in Landau frame : temperature and velocity have first principle definition

$$T^{\mu\nu} = \hat{\varepsilon}\left[\hat{u}^{\mu}\hat{u}^{\nu} + \frac{1}{3}\hat{\Delta}^{\mu\nu}\right] + \hat{\pi}^{\mu\nu}$$

The price is paid : (ii) or by introducing new 'non-fluid' variables as in MIS

$$\begin{aligned}
(1+\tilde{\theta}\hat{D})\hat{\pi}^{\mu\nu} &= -2\eta\hat{\sigma}^{\mu\nu} + \rho_1^{\mu\nu} \\
(1+\tilde{\chi}\hat{D})\rho_1^{\mu\nu} &= (-2\eta)(-\tilde{\theta})\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{T} + \rho_2^{\mu\nu} \\
(1+\tilde{\theta}\hat{D})\rho_2^{\mu\nu} &= (-2\eta)(-\tilde{\theta})\left(-\frac{\tilde{\chi}}{3}\right)\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}\cdot\hat{u} + \rho_3^{\mu\nu} \\
(1+\tilde{\chi}\hat{D})\rho_3^{\mu\nu} &= (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)\frac{1}{\hat{T}}\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{T} + \rho_4^{\mu\nu} \\
(1+\tilde{\theta}\hat{D})\rho_4^{\mu\nu} &= (-2\eta)(-\tilde{\theta})^2\left(-\frac{\tilde{\chi}}{3}\right)^2\hat{\nabla}^{\langle\mu}\hat{\nabla}^{\rangle\nu}\hat{\nabla}^2\hat{\nabla}\cdot\hat{u} + \cdots \\
&\vdots
\end{aligned}$$

Relaxation time scales are provided by the poles of the infinite sum of temporal derivatives

To maintain causality and stability we have to give up at least one of (i), (ii), (iii)

S Bhattacharyya, S Mitra and S Roy, Phys. Lett. B 856 (2024), 138918

Acausality in truncated 'MIS' theory

The all order information (over temporal derivative) is required for causality

MIS theory - an all order gradient correction theory

MIS theory :

$$\partial_\mu T^{\mu\nu} = 0 \ , \quad T^{\mu\nu} = \varepsilon\left(u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu}\right) + \pi^{\mu\nu}$$

$$\pi^{\mu\nu} + \tau_\pi D\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$

What if we demand the theory to be finitely truncated

$$\pi^{\mu\nu} = \sum_n \pi_n^{\mu\nu} \ , \qquad \pi_1^{\mu\nu} = -2\eta\sigma^{\mu\nu} \ , \qquad \pi_n^{\mu\nu} = -\tau_\pi D\pi_{n-1}^{\mu\nu} \ , n \geq 2$$

$$\pi^{\mu\nu} = -2\eta\left\{\sum_{n=0}^{N}\left(-\tau_\pi D\right)^n\right\}\sigma^{\mu\nu}$$

only with $N \to \infty$ and $|\tau_\pi D| < 1$

$$\pi^{\mu\nu} = -2\eta\left(1 + \tau_\pi D\right)^{-1}\sigma^{\mu\nu}$$

Acausality in truncated 'MIS' theory

Linear stability and causality analysis : $\psi(t,x) = \psi_0 + \delta\psi(t,x)$

Fluctuations are expressed in plane wave solutions via a Fourier transformation : $\delta\psi(t,x) \to e^{i(kx-\omega t)}\delta\psi(\omega,k)$

shear channel dispersion relation : $(i\omega) + \tilde{\eta}(ik)^2 \left[\sum_{n=0}^{N} (\tau_\pi i\omega)^n\right] = 0$

sound channel dispersion relation : $(i\omega)^2 + \frac{4}{3}\tilde{\eta}(i\omega)(ik)^2 \left[\sum_{n=0}^{N} (\tau_\pi i\omega)^n\right] - \frac{1}{3}(ik)^2 = 0$

Violates relativistic quantum theory causality condition $\mathrm{Im}(\omega(k)) \leq |\mathrm{Im}(k)|$ for any finite N

S Mitra, Phys. Rev. D 109 (2024) no.12, L121501

$$N \to \infty$$

shear channel : $\tau_\pi\omega^2 + i\omega - \tilde{\eta}k^2 = 0$

sound channel : $\tau_\pi\omega^3 + i\omega^2 - \left(\frac{4}{3}\tilde{\eta} + \frac{1}{3}\tau_\pi\right)\omega k^2 - \frac{1}{3}ik^2 = 0$

Respects relativistic quantum theory causality condition $\mathrm{Im}(\omega(k)) \leq |\mathrm{Im}(k)|$

Abide by the conservation rule of fluid modes : $\mathcal{O}_\omega[F(\omega, \boldsymbol{k} \neq 0)] = \mathcal{O}_{|\boldsymbol{k}|}[F(\omega = a|\boldsymbol{k}|, \boldsymbol{k} = \mathbf{b}|\boldsymbol{k}|]$

$\downarrow$

Condition for causality

Ref: Hoult and Kovtun, PRD 109 (2024) no.4, 046018

Newer degrees of freedom - a microscopic derivation

Probing physical origin of the newly promoted 'non-fluid' degrees of freedom - a kinetic theory perspective

- The underlying microscopic theories are always known to be free from pathologies concerning subluminal signal propagation.
- Then the process of coarse-graining must have to do something with the rising causality related issues.
- The truncation scheme could be the answer ?

Boltzmann transport equation : $\tilde{p}^\mu \partial_\mu f = -\frac{\tilde{E}_p}{\tau_R}\delta f = -\frac{\tilde{E}_p}{\tau_R}(f - f_0)$

Momentum distribution correction : $$\delta f = -\frac{\left[\frac{\tau_R}{\tilde{E}_p}\tilde{p}^\mu \partial_\mu\right] f_0}{\left[1 + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^\mu \partial_\mu\right]} = \frac{\frac{\tau_R}{\tilde{E}_p} f_0 \left[\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu} - \frac{1}{(\varepsilon+P)}\tilde{E}_p \tilde{p}_{\langle\nu\rangle}\nabla_\rho \pi^{\nu\rho}\right]}{\left[1 + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^\lambda \partial_\lambda\right]}$$

Exact expression of shear viscous flux (no truncation so far)

$$\frac{\pi^{\alpha\beta}}{T^2} = \int dF_p \frac{\frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu}}{\left[1 + \tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda\right]} - \frac{\tau_R}{(\varepsilon+P)}\int dF_p \frac{\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}_{\langle\mu\rangle}\nabla_\nu \pi^{\mu\nu}}{\left[1 + \tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda\right]}$$

Newer degrees of freedom - a microscopic derivation

Hydrodynamics - systematic build up of gradients of the fluid variables

$$\tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda \quad \sim \quad \frac{\lambda_{\rm mfp}}{L} = K_n$$

K_n = Knudsen number : decides the region of validity of the resulting coarse-grained (hydrodynamic) theory

$K_n < 1 \quad \sim \quad |\tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda| < 1$: limit for hydro to be valid in terms of systematic build up of gradients

- $|\tau_R D + \frac{\tau_R}{\tilde{E}_p}\tilde{p}^{\langle\lambda\rangle}\nabla_\lambda|$ is expanded as an infinite derivative sum series operating on thermodynamic forces.
- The infinite sum over the temporal derivatives forms a closed structure that creates relaxation operator like forms $(1+\tau_R D)$ in the denominator.

$$\frac{\pi^{\alpha\beta}}{T^2} = \sum_{m=0}^{\infty}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2m+1}\left[\int \frac{dF_p}{\left(\tilde{E}_p\right)^{2m+1}}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\tilde{p}^{\langle\lambda_1\rangle}\cdots\tilde{p}^{\langle\lambda_{2m}\rangle}\right]\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2m}}\sigma_{\mu\nu}$$
$$+\frac{1}{(\varepsilon+P)}\sum_{n=0}^{\infty}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2n+2}\left[\int \frac{dF_p}{\left(\tilde{E}_p\right)^{2n+1}}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}_{\langle\mu\rangle}\tilde{p}^{\langle\lambda_1\rangle}\cdots\tilde{p}^{\langle\lambda_{2n+1}\rangle}\right]\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+1}}\nabla_\nu\pi^{\mu\nu}$$

In each term the order of spatial gradient in the numerator exactly agrees with the order of temporal derivative in the denominator $\rightarrow$ **causality**

Newer degrees of freedom - a microscopic derivation

Exact expression of shear viscous flux (no truncation done)

$$\frac{\pi^{\alpha\beta}}{4!P}=\sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m+5)}\frac{1}{(2m+3)}\frac{1}{(2m+1)}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2m+1}\Big[\left(\nabla^2\right)^m\sigma^{\alpha\beta}+(4m)\left(\nabla^2\right)^{m-1}\nabla^{\langle\alpha}\nabla^{\nu}\sigma_{\nu}^{\beta\rangle}$$
$$+(2m)(m-1)\left(\nabla^2\right)^{m-2}\nabla^{\langle\alpha}\nabla^{\beta\rangle}\nabla_{\langle\mu}\nabla_{\nu\rangle}\sigma^{\mu\nu}\Big]$$
$$+\frac{1}{(\varepsilon+P)}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+5)}\frac{1}{(2n+3)}\left\{\frac{\tau_R}{1+\tau_R D}\right\}^{2n+2}\Big[\left(\nabla^2\right)^n\nabla^{\langle\alpha}\nabla^{\nu}\pi_{\nu}^{\beta\rangle}+n\left(\nabla^2\right)^{n-1}\nabla^{\langle\alpha}\nabla^{\beta\rangle}\nabla_{\langle\mu}\nabla_{\nu\rangle}\pi^{\mu\nu}\Big]$$

The nonlocal set of equations can be recast into a local set of equations by 'integrating in' new 'non-fluid' variables $\Longrightarrow$ **new degrees of freedom**

$$(1+\tau_R D)\,\pi^{\alpha\beta}=2\eta\sigma^{\alpha\beta}+\rho_1^{\alpha\beta}$$
$$(1+\tau_R D)^2\,\rho_1^{\alpha\beta}=-\eta\tau_R^2\left[\frac{2}{7}\nabla^2\sigma^{\alpha\beta}+\frac{12}{35}\nabla^{\langle\alpha}\nabla^{\nu}\sigma_{\nu}^{\beta\rangle}\right]+\rho_2^{\alpha\beta}$$
$$(1+\tau_R D)^2\,\rho_2^{\alpha\beta}=\eta\tau_R^4\left[\frac{2}{21}\nabla^4\sigma^{\alpha\beta}+\frac{32}{105}\nabla^2\nabla^{\langle\alpha}\nabla^{\nu}\sigma_{\nu}^{\beta\rangle}+\frac{4}{105}\nabla^{\langle\alpha}\nabla^{\beta\rangle}\nabla_{\langle\mu}\nabla_{\nu\rangle}\sigma^{\mu\nu}-\frac{16}{35}\Delta^{\alpha\beta}_{ab}\Delta^{b\nu}_{cd}\nabla^a\nabla^c\nabla_\nu\nabla_\rho\sigma^{\rho d}\right]+\rho_3^{\alpha\beta}$$
$$\vdots$$

S Mitra, Phys. Lett. B 860 (2025), 139174

For each higher order of spatial gradient truncation a new degree of freedom needs to be introduced
Otherwise, the causality of the theory is bound to be compromised

Conclusion

Summary···

- Causality and stability require a consensus between the defining fluid variables, their gauge fixing and the theory to be truncated - so far at least one has to be compromised.
- The causality related issues arise because of the gradient truncation scheme of the hydro theory from the microscopic derivation.
- No truncated theory can be causal - the all order information retains either via new degrees of freedom or field redefinition.

Take home message ···

- Introduction of non-hydro features in hydro is inevitable.
- The enigma remains how efficiently we can do that for an effective theory......

If we can not beat the riddles (at least fully), lets join the riddle......