Structure, Rotation, and Oscillation Properties of Anisotropic Dark Energy Stars

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Group

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- Results are presented in:
  - *Non-radial oscillations in anisotropic dark energy stars*;
    O. P. Jyothilakshmi, Lakshmi J. Naik, V. Sreekanth;
    Eur. Phys. J. C 84 (2024) 4, 427
  - *Aspects of rotating dark energy stars*;
    O. P. Jyothilakshmi, Lakshmi J. Naik, V. Sreekanth;
    Eur. Phys. J. C 84 (2024) 12, 1242

Introduction

- Dark energy is introduced to explain the accelerated expansion of the universe.
- Dark energy is found to fill up almost 70% of the universe based on various observations.
- There are various dynamical models that explain dark energy such as, cosmological constant, quintessence, K-essence, tachyon field, phantom field, dilatonic field, **Chaplygin gas model**.

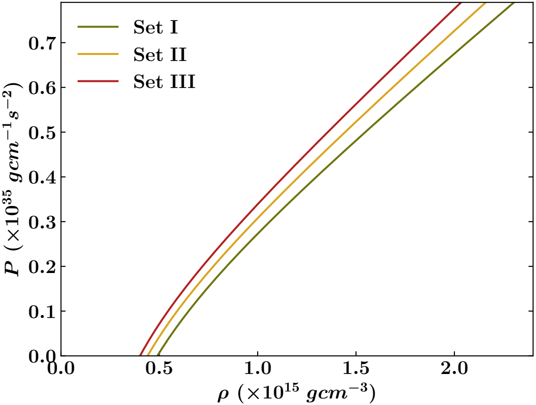
Dark Energy stars: Modified Chaplygin Model

- The fact that interior composition of a compact star is unknown to this date has motivated a lot of researchers to construct different models of compact stars.
- Dark energy star model was first proposed by Chapline in 2004, where the dark energy stars were considered to be an alternative to astrophysically observed black holes.
- We use a modified version of the Chaplygin prescription:

$$p = A^2\rho - \frac{B^2}{\rho}.$$

  Here, p is the pressure of the fluid, ρ is the energy density and A (no unit) and B (units of energy density) are positive constants.
- The first term represents a barotropic fluid and the second term corresponds to the Chaplygin gas.
- It must be noted that as the pressure vanishes at the surface of the star, the energy density there takes becomes $\rho_s = B/A$.

EOS



	Set I	Set II	Set III
A	$\sqrt{0.4}$	$\sqrt{0.425}$	$\sqrt{0.45}$
B (km^{-2})	0.23×10^{-3}	0.215×10^{-3}	0.2×10^{-3}

TABLE: The values of constants A and B of modified Chaplygin equation of state under consideration.

EOS

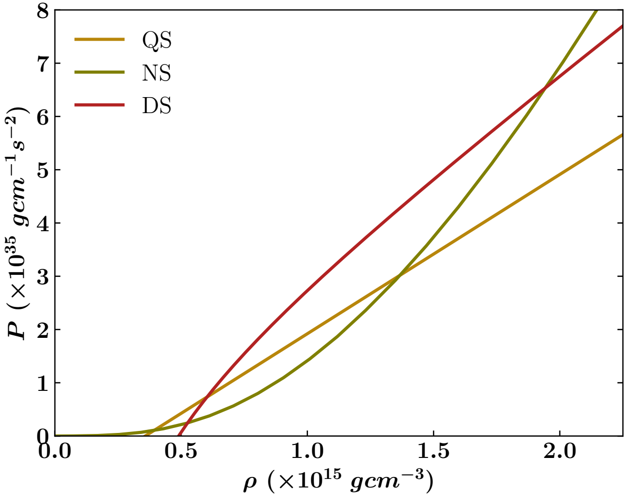


FIGURE: Chaplygin EoS (DS) plotted along with neutron star EoS (NS), with Sly4 EoS and quark star (QS), with modified Bag model EoS.

Anisotropy

- Generally, it is assumed that the stellar fluid is isotropic. However, the high densities and intense gravitational fields found in compact stars indicate that there may be pressure anisotropy within these objects.
- The stress-energy tensor $T^{\mu\nu}$ for an anisotropic fluid is given by

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + q g^{\mu\nu} + \sigma k^{\mu} k^{\nu};$$

  where $\sigma = p - q$ is the anisotropy pressure, p is the radial pressure, q is the tangential pressure. u^{μ} is the four-velocity of the fluid, and k_{ν} is the radial vector: $u_{\mu} u^{\mu} = -1$, $k_{\mu} k^{\mu} = 1$ and $u^{\mu} k_{\mu} = 0$.
- We use the Bowers-Liang model for anisotropy

$$\sigma = -\lambda_{BL} \frac{r^3}{3} \left(\rho(r) + 3p(r)\right) \left(\frac{\rho(r) + p(r)}{r - 2m(r)}\right);$$

  where free parameter λ_{BL} denotes the measure of anisotropy.

Stellar structure

- We consider the most general form of the metric of a static, spherically symmetric spacetime:

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

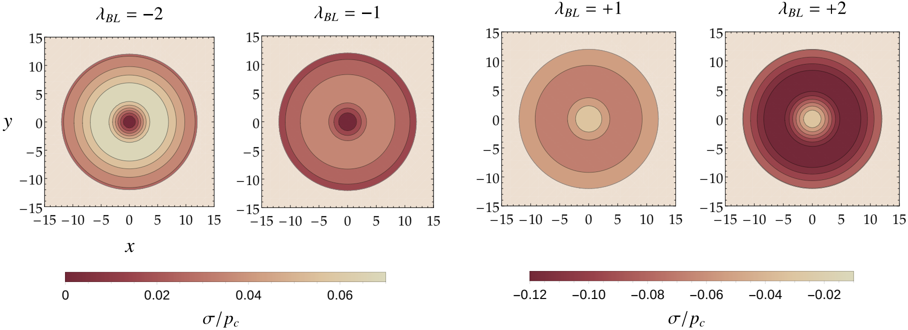
- With anisotropy this leads to modified Tolman-Oppenheimer-Volkoff (TOV) equations (coupled differential equations in the variables $p(r)$ and $m(r)$):

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r^2(1 - 2m/r)} - \frac{2\sigma}{r}, \quad \frac{dm}{dr} = 4\pi r^2 \varepsilon.$$

- The solution of the TOV equations for a specific EoS $\varepsilon = \varepsilon(p)$ is determined by two boundary conditions: $p_c = p(r = 0)$ and $m_c = m(r = 0) = 0$.

Anisotropic pressure profile

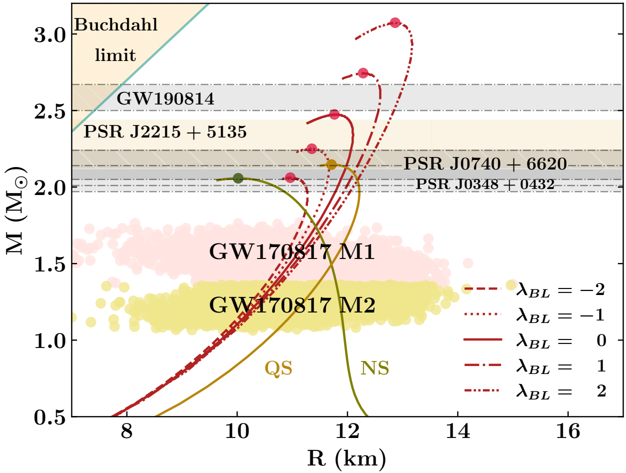
- We see that the anisotropic pressure increases from 0 ($\lambda_{BL} = 0$), reaches a maximum value and then slowly decreases with r for $|\lambda_{BL}|$ values.



(a) (b)

Results

- With a specified central density, the TOV equations can be integrated outwards starting at the center. The radius R of the star is defined as the distance at which the pressure drops to zero, i.e. $p(r = R) = 0$. The mass of the star is then given by $M = m(R)$.



- We find that the internal stellar structure is highly sensitive to the strength of anisotropy.
- Increasing λ_{BL} from -2 to $+2$ leads to higher maximum mass and radius.
- The mass–radius profiles of dark energy stars differ markedly from those of neutron stars, exhibiting qualitative features more akin to quark star configurations.

Non-radial Oscillations of compact stars

- Oscillations where fluid elements move in angular directions (θ, ϕ) in addition to the radial direction.
- Unlike radial oscillations, the motion is not spherically symmetric.
- Described using spherical harmonics $Y_{\ell m}(\theta, \phi)$.

Mode	Restoring Force	Features
f-mode	Pressure	Fundamental; No radial nodes in the fluid displacement
p-modes	Pressure	Higher frequency, radial overtones
g-modes	Buoyancy	Sensitive to stratification, low freq.
r-modes	Coriolis	Damping due to viscous effects
w-modes	Spacetime curvature	Relativistic; weak fluid motion

- The f-modes are the fundamental modes of oscillation. The f-mode frequencies are much lower than other modes. Thus they maybe detectable with the current generation of GW detectors.

Non-radial f-modes: Cowling Approximation

- Perturbations of a static spherically symmetric star ($g^B_{\mu\nu}$) is

$$g_{\mu\nu} = g^B_{\mu\nu} + h_{\mu\nu}.$$

- Metric perturbations ($h_{\mu\nu}$) are expanded in tensor spherical harmonics.
- The perturbed stress energy tensor:

$$\delta T_{\mu\nu} = (\delta\rho + \delta P)u_\mu u_\nu + (\rho + P)(\delta u_\mu u_\nu + u_\mu \delta u_\nu) + \delta P\, g_{\mu\nu} + P\, \delta g_{\mu\nu}$$

- Lagrange fluid displacement vector ($\boldsymbol{\xi}$) that represent the infinitesimal oscillatory perturbations of the star:

$$\boldsymbol{\xi}^i = \left(e^{-\Lambda}W,\ -V\partial_\theta,\ -V\sin^{-2}\theta\,\partial_\phi\right) r^{-2} Y_{\ell m} e^{i\omega t}.$$

- We aim to obtain the f-mode frequency using Cowling approximation.

Non-radial f-modes: Cowling Approximation

- Cowling approximation neglects the metric perturbation i. e., $h_{\mu\nu} = 0$
- We make use of Cowling approximation and obtain the coupled differential equation for anisotropic dark energy stars, in terms of the perturbation functions W and V.

$$\frac{dW}{dr} = \frac{d\rho}{dp}\Big[\omega^2 \frac{\rho + p - \sigma}{\rho + p}\left(1 - \frac{\partial\sigma}{\partial p}\right)^{-1} e^{\lambda - 2\Phi} r^2 V + \frac{d\Phi}{dr} W\Big] - l(l+1)e^{\lambda} V$$
$$+ \frac{\sigma}{\rho + p}\left[\frac{2}{r}\left(1 + \frac{d\rho}{dp}\right) W + l(l+1)e^{\lambda} V\right],$$
$$\frac{dV}{dr} = 2V\frac{d\Phi}{dr} - \left(1 - \frac{\partial\sigma}{\partial p}\right)\frac{\rho + p}{\rho + p - \sigma}\frac{e^{\lambda}}{r^2} W$$
$$+ \Big[\frac{\sigma\,'}{\rho + p - \sigma} + \left(\frac{d\rho}{dp} + 1\right)\frac{\sigma}{\rho + p - \sigma}\left(\frac{d\Phi}{dr} + \frac{2}{r}\right)$$
$$- \frac{2}{r}\frac{\partial\sigma}{\partial p} - \left(1 - \frac{\partial\sigma}{\partial p}\right)^{-1}\left(\frac{\partial^2\sigma}{\partial p^2} p' + \frac{\partial^2\sigma}{\partial p\partial\mu}\mu'\right)\Big] V.$$

- Towards the center of the star:

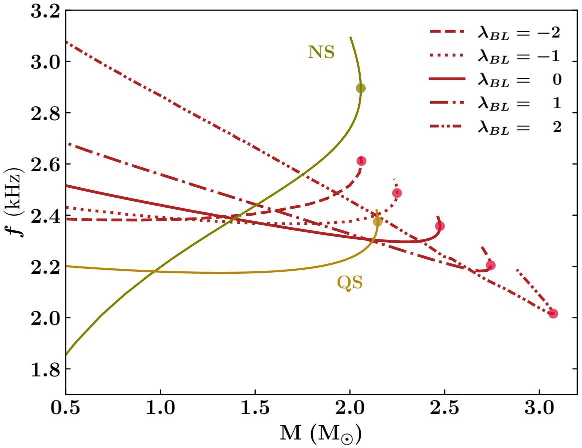
$$W = Ar^{l+1}, \quad V = -\frac{A}{l}r^{l};$$

- The coupled differential equations are solved from the center to the surface with an initial guess of ω. The value of ω is modified until the surface boundary condition is satisfied:

$$\omega^2\left(1-\frac{\sigma}{\rho+p}\right)\left(1-\frac{\partial\sigma}{\partial p}\right)^{-1}e^{-2\Phi}V+\left(\frac{d\Phi}{dr}+\frac{2}{r}\frac{\sigma}{\rho+p}\right)e^{-\lambda}\frac{W}{r^2}=0.$$

[D.D. Doneva, S.S. Yazadjiev, Phys. Rev. D 85, 124023 (2012).]

Results



[O. P. Jyothilakshmi, Lakshmi J. Naik, V. Sreekanth; Eur. Phys. J. C 84 (2024) 4, 427]

Hartle–Thorne slow rotation prescription

- The **Hartle–Thorne method** is a perturbative approach to model slowly rotating relativistic stars in general relativity.
- Developed by James Hartle (1967) and James Hartle & Kip Thorne (1968).
- This method assumes a universal rotation with angular velocity Ω, with Ω valid when $\Omega^2 R^3 \ll GM$. Also, metric, fluid variables, and spacetime geometry are expanded only upto $\mathcal{O}(\Omega^2)$.
- Results
  - Frame-dragging: spacetime develops an off-diagonal $g_{t\phi}$ term.
  - First-order: determines angular momentum J, moment of inertia I.
  - Second-order: includes shape deformation and quadrupole moment Q.

Stellar structure: Slow rotation

- The metric of a slowly rotating, axially symmetric equilibrium configuration can be written in the form:

$$ds^2 = -e^{2\nu}[1+2(h_0+h_2P_2)]dt^2 + \frac{r\left[1+2(m_0+m_2P_2)(r-2m)^{-1}\right]}{r-2m}dr^2$$
$$+ r^2[1+2(v_2-h_2)P_2]\left[d\theta^2+\sin^2\theta(d\phi-\omega dt)^2\right] + O(\Omega^3).$$

  Here, $P_2 = P_2(\cos\theta)$ is the second order Legendre polynomial and the perturbative terms h_0, m_0, h_2, m_2, p_2, and v_2 are functions of r and are proportional to the square of angular velocity of the star $\Omega^2(r)$.

- Here each metric potential ν, ψ, μ and λ depends on r, θ and Ω, and ω is the frame dragging frequency.

[James B. Hartle and Kip S. Thorne, Astrophys. J. 153, 807 (1968)]

The mass (M_{rot}) and radius (R_{rot}) of a slowly rotating star in the Hartle-Thorne formalism are given by

$$\begin{aligned} M_{rot} &= M + m_0(R) + \frac{J^2}{R^3}, \\ R_{rot} &= R + \xi(r,\theta) = R + \xi_0(R) + \xi_2(R)P_2. \end{aligned}$$

where, the spherical deformation factor

$$\xi_0(r) = -p_0^*(r)(\rho(r) + p(r))(dp/dr)^{-1}.$$

The quadrupole deformation ξ_2, which is expressed as

$$\xi_2(r) = -p_2 \left(\rho + p - \sigma\right) \left(\frac{dp}{dr} - \frac{d\sigma}{dr}\right)^{-1};$$

where, $p_2 = -(h_2 + \frac{1}{3}e^{-2\nu}r^2\bar{\omega}^2)$ is the quadrupole pressure perturbation factor.

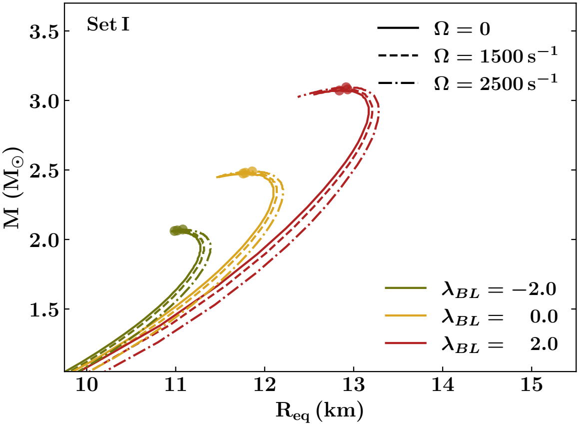


FIGURE: O. P. Jyothilakshmi, L. J. Naik, V. Sreekanth; Eur. Phys. J. C 84 (2024) 12, 1242

- Rotation adds centrifugal support, allowing the star to remain stable at higher mass and larger radius before collapsing.
- The maximum mass and radius increase for positive and decrease for negative λ_{BL}, consistent with the static case.

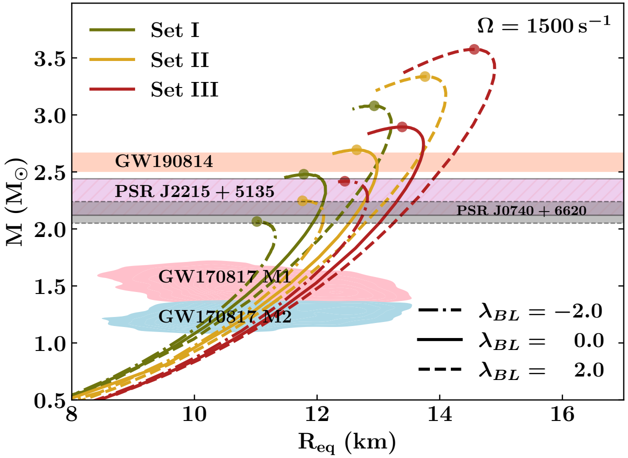


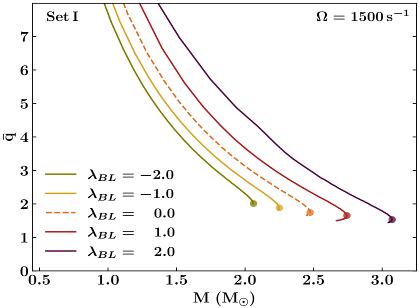
FIGURE: O.P. Jyothilakshmi, L. J. Naik, V. Sreekanth; Eur. Phys. J. C 84 (2024) 12, 1242

We note that for different values of λ_{BL}, Set III gives the highest maximum mass, since a stiffer EoS results in higher maximum mass.

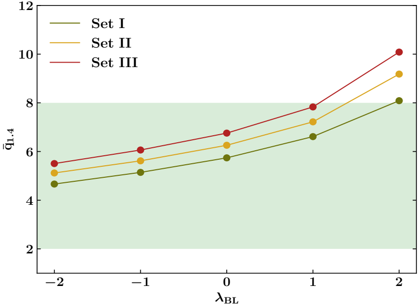
- We obtain the mass quadrupole moment Q of a rotating star given by

$$Q = \frac{8}{5}\mathcal{K}_2 M^3 + \frac{J^2}{M}.$$

- The quadrupole moment Q can be written with respect to the Kerr solution as a dimensionless quantity, $\bar{q} = QM/J^2$ and can be used to describe the deviation of the Hartle-Thorne metric from the Kerr black hole metric, for which the Kerr factor tends to unity.



(a) The Kerr factor $\bar{q} = QM/J^2$ as a function of stellar mass M of anisotropic dark energy stars obeying EoS Set I



(b) $\bar{q}_{1.4}$ for different values of λ_{BL} for three different parameter sets.

Conclusions

- We obtained the f-mode spectra for both isotropic and anisotropic dark energy stars using the Cowling approximation.
- We found that the dark energy stars exhibit a distinct behaviour compared to both neutron star and quark star.
- We observe that dark energy stars with higher anisotropic strength tend to approach the Kerr solution ($q = 1$).

THANK YOU

Chaplygin model

- *Chaplygin gas* obeying the equation of state (EoS)

$$p = -\hat{B}/\rho^{\omega}, \tag{1}$$

- Furthermore, a modified Chaplygin EoS was later constructed by including the effects of viscosity

$$p = \hat{A}\rho - \frac{\hat{B}}{\rho^{\omega}}, \tag{2}$$

where $\hat{A}$ is a dimensionless positive constant. The first term represents a barotropic and the second term corresponds to the Chaplygin gas. By setting the value of $\omega = 1$, one obtains the modified Chaplygin EoS in the form

$$p = A^2\rho - \frac{B^2}{\rho}. \tag{3}$$

Since the pressure vanishes at the surface of the star, the energy density towards the surface takes the value of $\rho_s = B/A$.