Two-loop Feynman integrals for tty hadron production in the leading color approximation

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In Collaboration with:
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based on arXiv:2610.42062

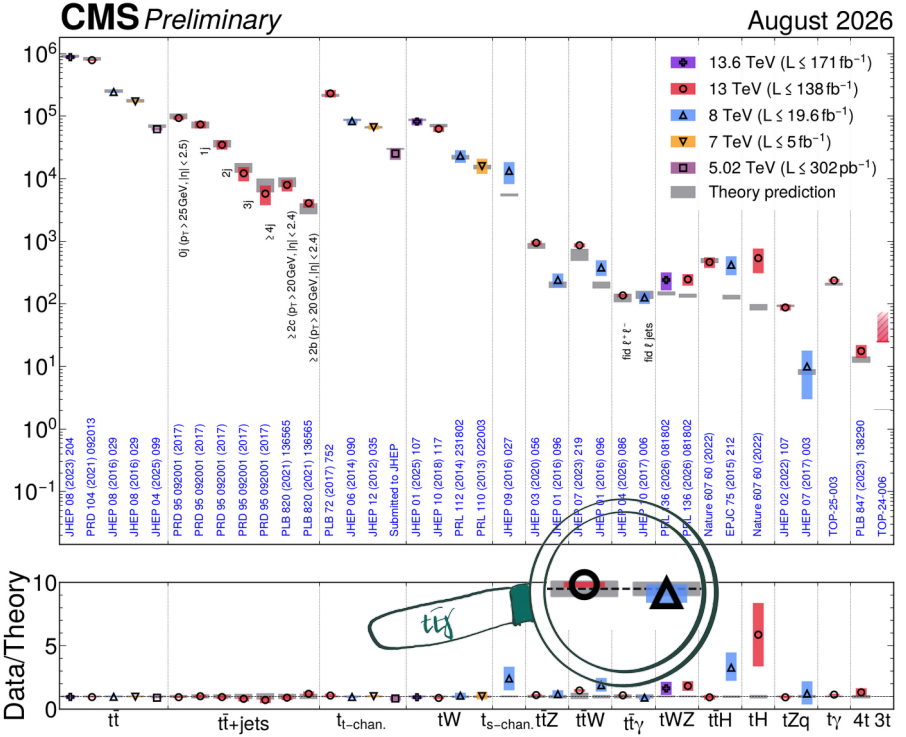
Online Seminar @ NISER, Bhubaneswar, India,
20th November 2026

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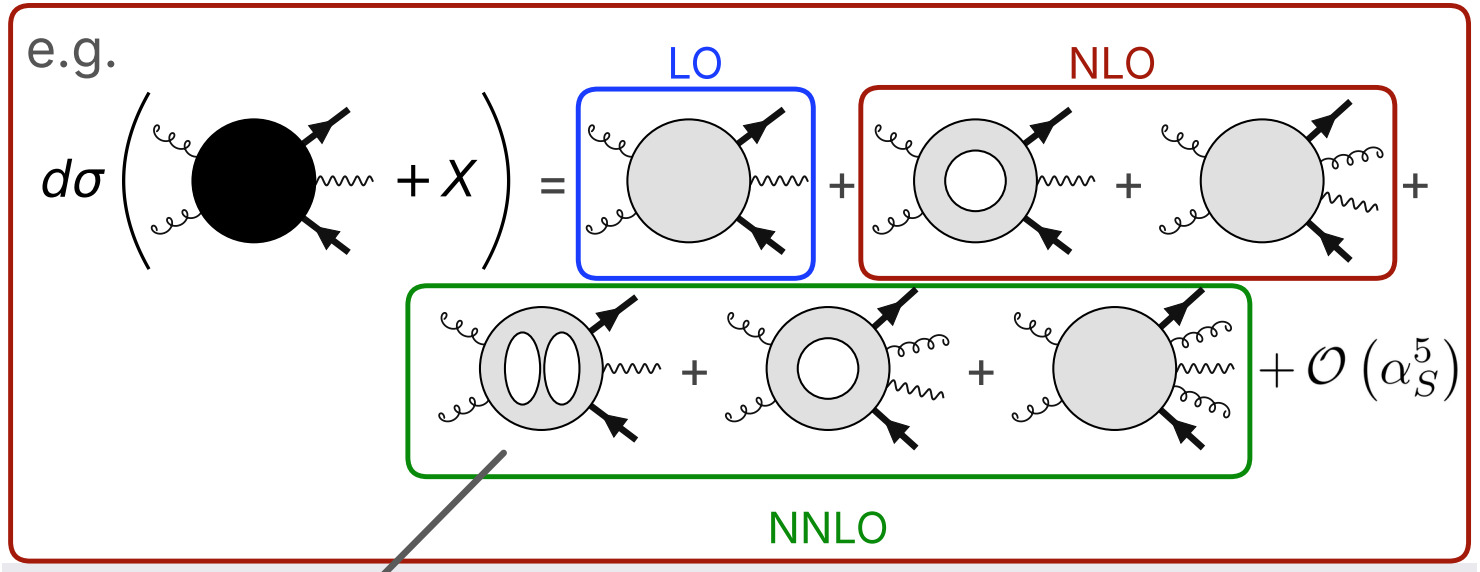
Precision physics with top quarks

- **pp → tt̄** and **pp → tt̄+X** (X=j,γ,Z,W,H,tt̄,...)
  are crucial processes for the LHC precision program
  - precision **tests of the SM**
  - measurement of key SM parameters
    m_t, α_s, y_t
  - Initial state PDFs determination
- tt̄γ production
  - direct measurement of the **top-quark electric charge**
  - Structure of the photon coupling with top quark



As **experimental precision grows**, theory prediction must keep pace

Theory challenges at NNLO



Two-loop amplitude (often missing)

$$\mathcal{A} = C.I$$

IR subtraction methods

STRIPPER, Antenna, nested-soft,...

$pp \to t\bar{t}\gamma$ theory status

Fixed order & matching

- NLO QCD and EW [Duan, Ma, Zhang, Han, Guo, Wang, Song, Li, 2009, 2011, 2017; Maltoni, Pagani, Tsinikos, 2016]
- Complete NLO corrections [Pagani, Shao, Tsinikos, Zaro, 2021]
- NLO QCD matched with the Pythia parton shower program [Kardos, Trócsányi, 2015]
- Top decays: narrow-width [Melnikov, Schulze, Scharf, 2011; Bevilacqua, Hartanto, Kraus, Weber, Worek, 2020, Stremmer, Worek, 2024]
- Top decays: full off-shell [Bevilacqua, Hartanto, Kraus, Weber, Worek, 2018, 2019; Stremmer, Worek, 2025]

Towards exact NNLO

- Approx. NNLO cross-section: soft-gluon corrections added to the complete NLO result [Kidonakis, Tonero, 2023]
- Approx. two-loop amplitudes: massification procedure [Wang, Xia, Yang, Ye, 2024]
- One-loop amplitudes up to $O(\varepsilon^2)$ [Bera, Brancaccio, Canko, Hartanto, 2025]

The next frontier: NNLO QCD accuracy for $pp \to t\bar{t}\gamma$

The main bottleneck: two-loop Feynman integrals and scattering amplitudes

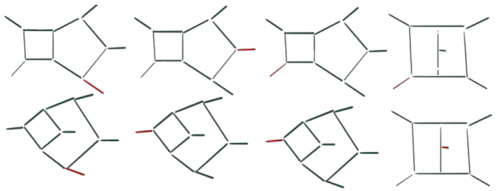
Two-loop five-point integrals - no internal masses

Massless external legs



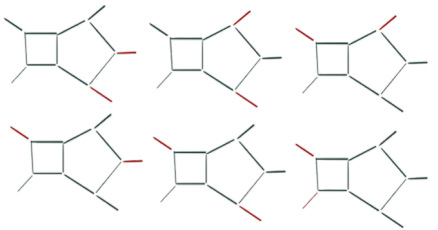
[Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, Lo Presti, Papadopoulos, Page, Sotnikov, Tomassini, Wasser, Wever, Zeng, Zhang, Zoia...]

One massive external leg



[Abreu, Canko, Chicherin, Ita, Kardos, Moriello, Page, Papadopoulos, Smirnov, Sotnikov, Syrrakos, Tomassini, Tschernow, Wever, Zeng, Zoia...]

Two massive external legs (planar)



[Jiang, Liu, Xu, Yang, 2024;
Abreu, Chicherin, Sotnikov, Zoia, 2024]

Canonical dLog Differential equations. MIs expanded in terms of a **basis of special functions (pentagon functions)**, which enables:

[Gehrmann, Henn, Lo Presti, 2018]
[Chicherin, Sotnikov, 2020]
[Chicherin, Sotnikov, Zoia, 2022]
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2024]

- analytic ε-pole cancellations;
- simplifications at amplitude level;
- fast and stable numerical evaluation.

Many amplitude and cross-section results available for processes sharing these topologies:

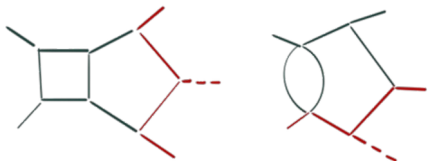
pp → 3j, pp → 3γ, pp → γγj, pp → γjj, pp → W+2j, pp → Wbb, pp → Hbb, pp → Wγγ, pp → H+2j, e+e- → 4j

[Abreu, Agarwal, Badger, Buccioni, Chawdhry, Chen, Chicherin, Czakon, De Laurentis, Febres-Cordero, Fox, Gambuti, Gehrmann, Glover, Henn, Ita, Lo Presti, Klinkert, Kuschke, Manteuffel, Ma, Marcoli, Mitov, Page, Peraro, Poncelet, Ruf, Schabinger, Sotnikov, Sun, Tancredi, Zhang, Zoia...]

Two-loop five-point integrals – massive propagators

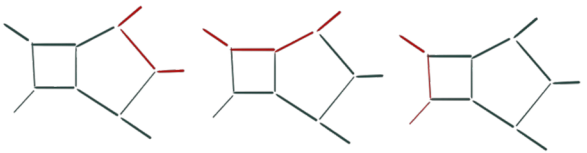
Increasing **analytic complexity** → presence of complicated (elliptic) geometries

pp → ttH – leading colour, N_f part



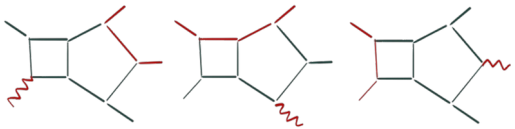
[Febres Cordero, Figueiredo, Kraus, Page, Reina, 2023]

pp → ttj – leading colour



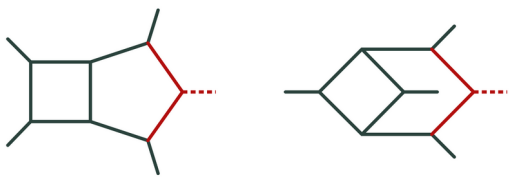
[Badger, Becchetti, Chaubey, Marzucca, 2022;
Badger, Becchetti, Giraudo, Zoia, 2024;
Becchetti, Dlapa, Zoia, 2025]

pp → ttW – leading colour



[Becchetti, Canko, Chestnov,
Peraro, Pozzoli, Zoia, 2025]

VBF @ NNLO QCD

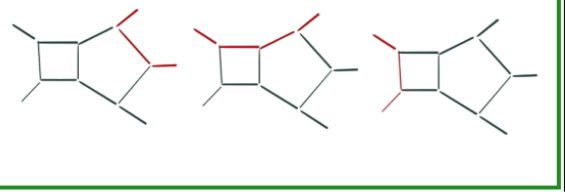
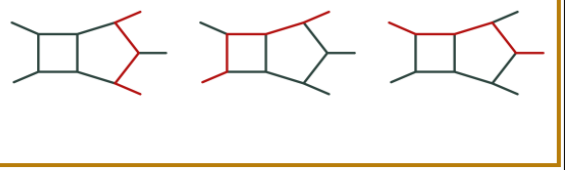
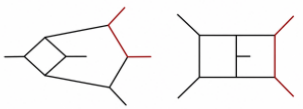
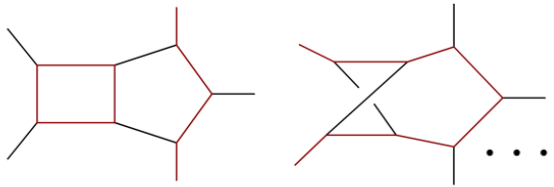


[Gotzfried, von Manteuffel, 2026]

Few NNLO QCD cross-section results for processes involving these topologies are available.
Limited to the leading-color approximation (or to other approximations of the double virtual contribution)

[Badger, Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, Zoia, 2025]
[Becchetti, Canko, Chen, Chestnov, Delto, Ditsch, Peraro, Pozzoli, Savoini, Zoia, 2026]
[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini, 2025]
[Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini, 2025]

$t\bar{t}y$ in generalised color expansion

$N_c^2 Q_q,\ N_c N_f Q_q,\ N_f^2 Q_q$	
$N_c^2 Q_t,\ N_c N_f Q_t,\ N_f^2 Q_t$	
$N_c \sum_f Q_f$	
sub-leading color, N_h contribution	

→ Recyclable from tt+jet leading color

→ New result of this work!

We take only the leading term in the amplitude when Nc and Nf are both large

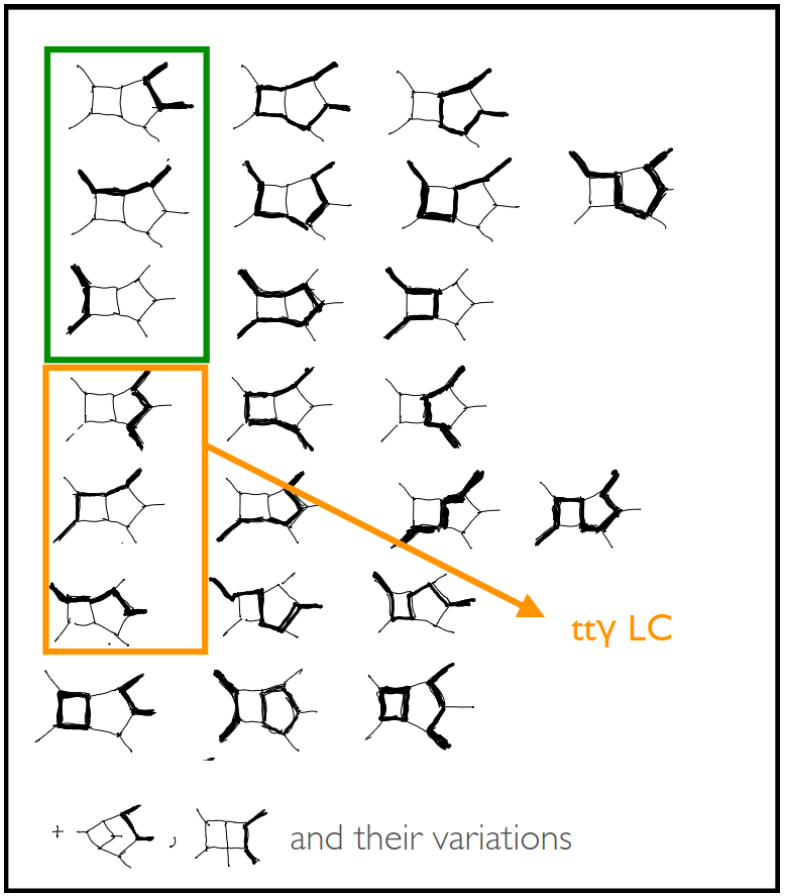
Sub-leading terms are extremely complicated, complicated geometries, algebraically challenging

Two-loop corrections to tt+jet

- 3-4% uncertainty at high pT or invariant mass from missing sub-leading color.
  Possibly sub-leading color effects also at low invariant mass, where there is a strong sensitivity to higher-order corrections.
  [Badger, Becchetti, Brancaccio, Czakon, Hartanto, Poncelet, Zoia, 2025]
- The full colour decomposition would require a large number of integral topologies each with ‘complicated geometries’
- Leading colour requires only three

c.f. 5-point massless with 1 external scale (Hjj) has 8 topologies for full colour

[Badger QCD@LHC26]



New Two-Loop Planar Topologies for Leading Colour (LC) tty

$$I^{(\mathrm{F})}_{\vec{\nu}} = \int \frac{\mathrm{d}^d k_1\, e^{\varepsilon\gamma_{\mathrm{E}}}}{\mathrm{i}\pi^{\frac{d}{2}}} \frac{\mathrm{d}^d k_2\, e^{\varepsilon\gamma_{\mathrm{E}}}}{\mathrm{i}\pi^{\frac{d}{2}}} \frac{D_{\mathrm{F},9}^{-\nu_9} D_{\mathrm{F},10}^{-\nu_{10}} D_{\mathrm{F},11}^{-\nu_{11}}}{D_{\mathrm{F},1}^{\nu_1} D_{\mathrm{F},2}^{\nu_2} D_{\mathrm{F},3}^{\nu_3} D_{\mathrm{F},4}^{\nu_4} D_{\mathrm{F},5}^{\nu_5} D_{\mathrm{F},6}^{\nu_6} D_{\mathrm{F},7}^{\nu_7} D_{\mathrm{F},8}^{\nu_8}} \qquad d = 4 - 2\varepsilon$$

F	PB_D	PB_E	PB_F
$D_{\mathrm{F},1}$	k_1^2	$k_1^2 - m_t^2$	$k_1^2 - m_t^2$
$D_{\mathrm{F},2}$	$(k_1 - p_1)^2 - m_t^2$	$(k_1 - p_1)^2$	$(k_1 - p_3)^2 - m_t^2$
$D_{\mathrm{F},3}$	$(k_1 - p_{13})^2 - m_t^2$	$(k_1 + p_{234})^2$	$(k_1 - p_{13})^2$
$D_{\mathrm{F},4}$	$(k_1 - p_{123})^2$	$(k_1 + p_{23})^2$	$(k_1 + p_{24})^2$
$D_{\mathrm{F},5}$	k_2^2	$k_2^2 - m_t^2$	$k_2^2 - m_t^2$
$D_{\mathrm{F},6}$	$(k_2 + p_{123})^2$	$(k_2 - p_3)^2 - m_t^2$	$(k_2 - p_2)^2$
$D_{\mathrm{F},7}$	$(k_2 + p_{1234})^2$	$(k_2 - p_{23})^2$	$(k_2 - p_{24})^2$
$D_{\mathrm{F},8}$	$(k_1 + k_2)^2$	$(k_1 + k_2)^2$	$(k_1 + k_2)^2$
$D_{\mathrm{F},9}$	$(k_1 - p_{1234})^2$	$(k_1 + p_3)^2 - m_t^2$	$(k_1 + p_2)^2$
$D_{\mathrm{F},10}$	$(k_2 + p_1)^2 - m_t^2$	$(k_2 + p_1)^2$	$(k_2 + p_3)^2 - m_t^2$
$D_{\mathrm{F},11}$	$(k_2 + p_{13})^2 - m_t^2$	$(k_2 - p_{234})^2$	$(k_2 + p_{13})^2$

$p_{i\cdots j} = p_i + \cdots + p_j$

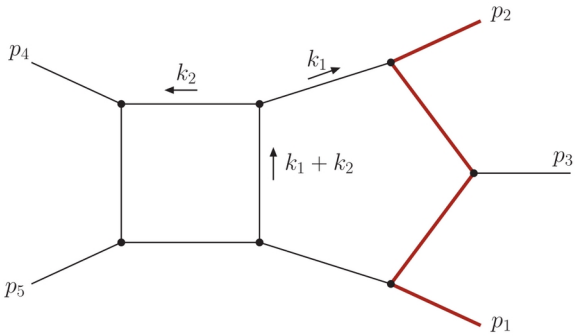
$I^{(\mathrm{F})}_{\vec{\nu}}$ form a finite vector space

The elements of a basis of integrals are also called **Master Integrals (MIs)**

Reduction to a basis by means of **Integration-by-part identities (IBPs)**

New Two-Loop Planar Topologies for Leading Colour (LC) tty

PBD



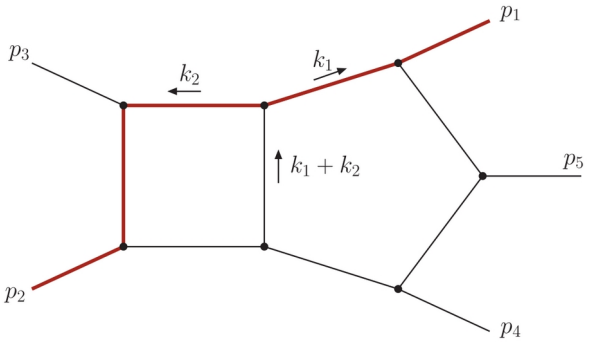
[Febres Cordero,Figueiredo,Kraus,Page,Reina ‘24]

100 MIs

Canonical basis ✓ [Henn ‘13]

d log alphabet

PBE



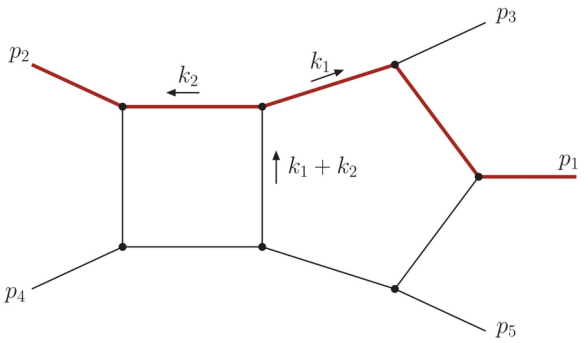
154 MIs

Canonical basis ✗

2 Elliptic sectors

1 Nested square-root (NS)

PBF



142 MIs

Canonical basis ✗

3 Elliptic sectors

2 NS

Differential equations: general structure

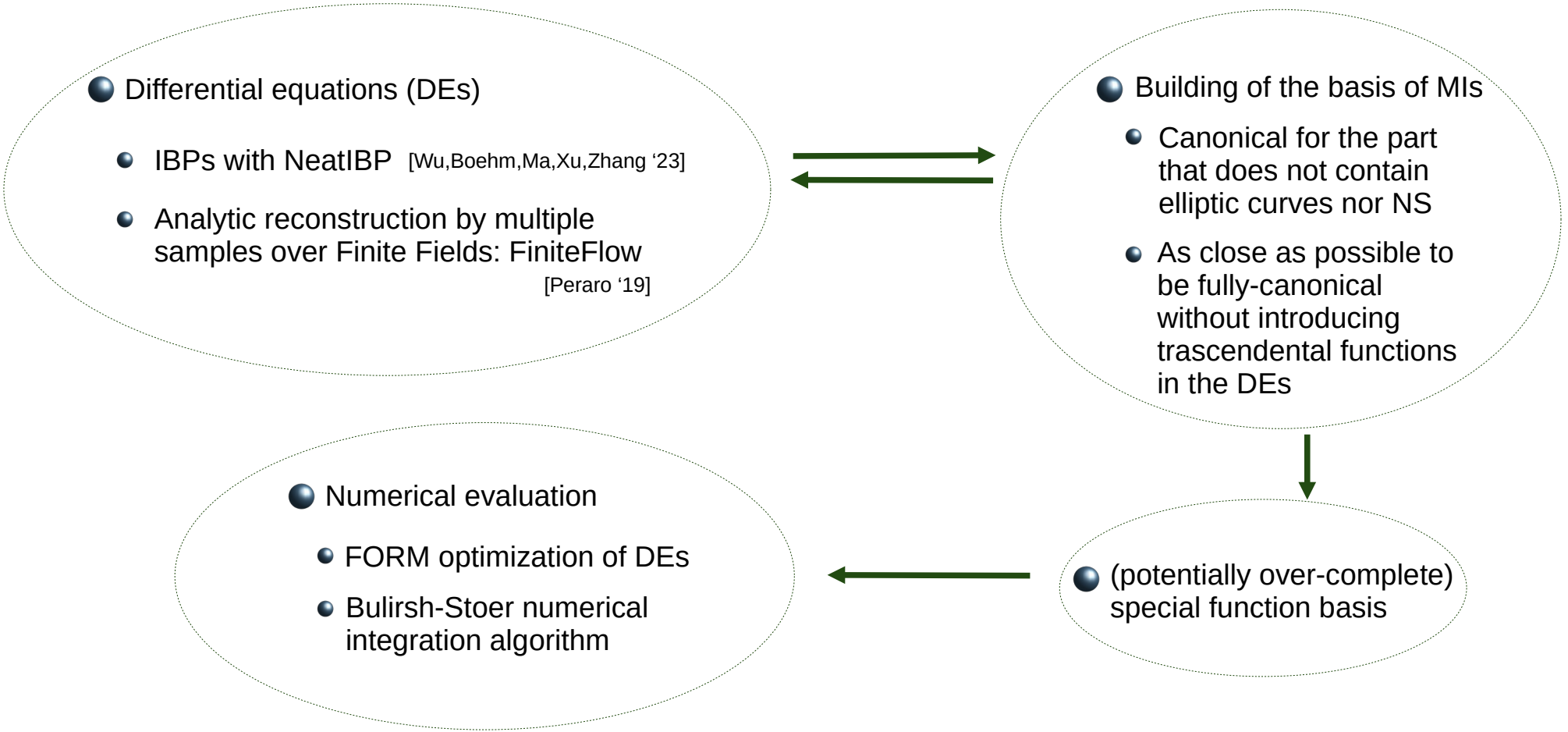
$$\frac{\partial}{\partial \xi}\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) = \mathcal{A}_{\xi}^{(\mathrm{F})}(\vec{x},\varepsilon)\cdot\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon)$$

- Kinematics variables $\vec{x} = \left(d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2\right) \quad d_{ij} = p_i \cdot p_j$
- $\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon)$ Basis of integrals
- $\mathcal{A}_{\xi}^{(\mathrm{F})}(\vec{x},\varepsilon)$ Connection matrices

The solution has not essential singularities (unphysical). So there is always a transformation of basis such that the connection matrix has only simple poles.

$$\tilde{\mathcal{A}}^{(\mathrm{F})}\left(\vec{x}(\lambda),\varepsilon\right) \sim \frac{a_k(\varepsilon)}{\lambda-\lambda_k} \quad \lambda \sim \lambda_k \qquad \forall\, \lambda_k \text{ singular point}$$

Strategy of the Computation



Differential equation structure

Canonical DEs: PBD

$$\mathrm{d}\mathcal{A}^{(\mathrm{PBD})}(\vec{x},\varepsilon) = \varepsilon \sum_i \alpha_i \,\mathrm{d}\log W_i(\vec{x})$$

ε-polynomial DEs: PBE, PBF

$$\mathrm{d}\mathcal{A}^{(\mathrm{F})}(\vec{x},\varepsilon) = \sum_i \left(\beta_{0i} + \beta_{1i}\varepsilon + \beta_{2i}\varepsilon^2\right)\omega_i(\vec{x}) + \left(\gamma_{0i} + \gamma_{1i}\varepsilon + \gamma_{2i}\varepsilon^2\right)\mathrm{d}\log W_i(\vec{x})$$

- $W_i(\vec{x})$ Letters: rational and algebraic
- Algebraic letters have the form: $\frac{A+B\sqrt{C}}{A-B\sqrt{C}}$ found with the help of the algorithm described in [Matijasic '24 PhD Thesis]
- $\alpha_i, \gamma_{0i}, \gamma_{1i}, \gamma_{2i}, \beta_{0i}, \beta_{1i}, \beta_{2i}$ Matrices of rational numbers
- $\omega_i(\vec{x})$ Linearly independent one-forms

  Algebraic one-forms: $\omega_i = \sqrt{C_i}\,\tilde{\omega}_i$ with $\tilde{\omega}_i$ rational

topology	square roots	d log letters	one-forms
PBD	6	77	0
PBE	9	113	177
PBF	11	129	222
2-Loop LC Q_t part	13	167	389

Differential equation structure

Canonical DEs: PBD

$$\mathrm{d}\mathcal{A}^{(\mathrm{PBD})}(\vec{x},\varepsilon) = \varepsilon \sum_i \alpha_i \,\mathrm{d}\log W_i(\vec{x})$$

ε-polynomial DEs: PBE, PBF

$$\mathrm{d}\mathcal{A}^{(\mathrm{F})}(\vec{x},\varepsilon) = \sum_i \left(\beta_{0i} + \beta_{1i}\varepsilon + \beta_{2i}\varepsilon^2\right)\omega_i(\vec{x}) + \left(\gamma_{0i} + \gamma_{1i}\varepsilon + \gamma_{2i}\varepsilon^2\right)\mathrm{d}\log W_i(\vec{x})$$

- Properties of ε-polynomial DEs:
  - Canonical for blocks that do not couple to elliptic curves or NS
  - Non-canonical entries are polynomials, at worst quadratic, in ε (linear for NS)
  - In the Laurent expansion of MIs, elliptic or NS part starts at $O(\varepsilon^4)$ → Analytic cancellation of the poles in the amplitude

[Becchetti,Canko,Chestnov,Peraro,Pozzoli,Zoia '25]
[Badger,Becchetti,Giraudo,Zoia '24]

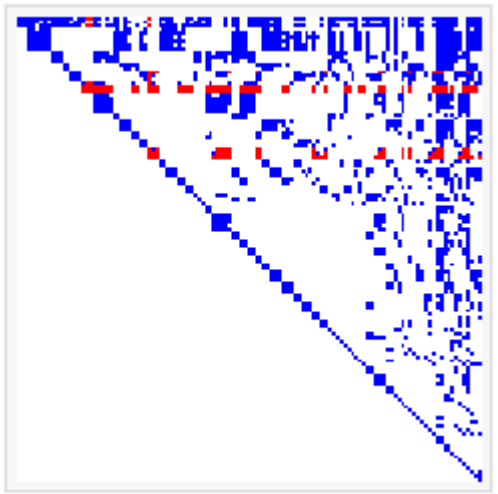
- Significant progress in defining and constructing canonical DEs beyond d Log: i) ε-factorised, ii) only simple poles, iii) one-forms independent up to total derivatives

  [Adams,Chaubey,Weinzierl '17] [Adams,Weinzierl '18] [Frellesvig '22] [Görges,Nega,Tancredi,Wagner '23] [Dlapa,Henn,Wagner '23] [Pögel,Wang,Weinzierl '23] [Pögel,Wang,Weinzierl '23] [Frellesvig,Weinzierl '24] [Driesse,Jakobsen,Klemm,Mogull,Nega,Plefka,Sauer,Usovitsch '25] [Duhr,Porkert,Stawinski '25] [Duhr,Maggio,Nega,Sauer,Tancredi,Wagner '25] [Chen,Yang,Zhang '25] [Forner,Mella,Nega,Tancredi,Wagner '26]

- We DO NOT provide fully canonical DEs

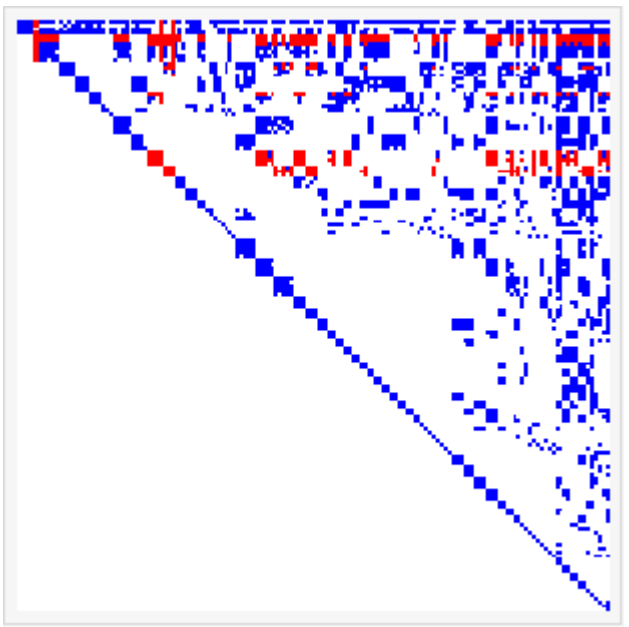
Differential equation structure

PBB 2 massive propagators

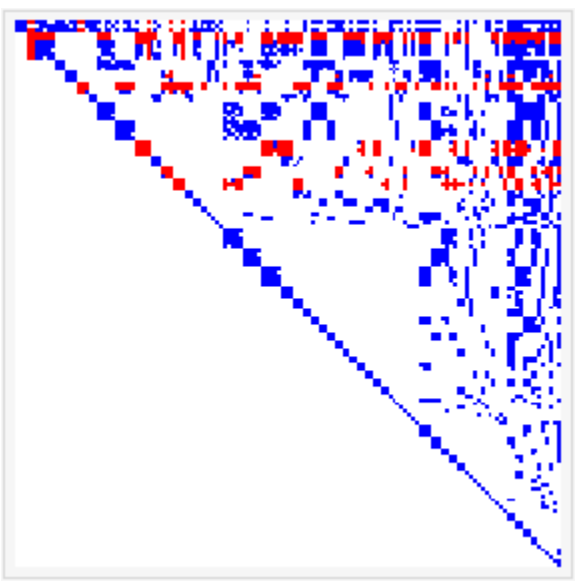


[Badger,Becchetti,Giraudo,Zoia ‘24]

PBE 3 massive propagators



PBF 3 massive propagators



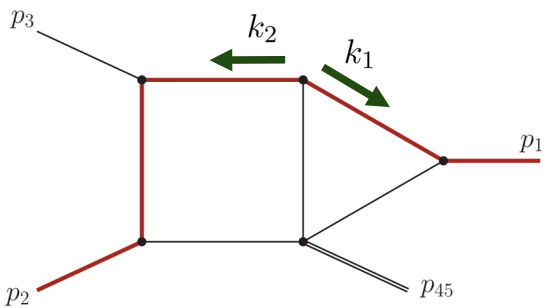
Masses make mathematical structure increasingly complex

Canonical ε d Log

Everything else

Elliptic sector $I^{\mathrm{PBE}}_{11001111000}$

- Basis of integrals (3 MIs)
  - simple pole: $\mathrm{MC}\left(I^{\mathrm{PBE}}_{11001111000}[k_2 \cdot p_1]\right)_{\varepsilon=0} \sim \int \frac{\mathrm{d}z\, z}{\sqrt{P_4(z)}} + (\text{holomorfic})$
  - double pole: $\mathrm{MC}\left(I^{\mathrm{PBE}}_{11001111000}[k_1 \cdot p_3]\right)_{\varepsilon=0} \sim \int \frac{\mathrm{d}z}{(z-4m_t^2)\sqrt{P_4(z)}}$
  - holomorfic: $\mathrm{MC}\left(I^{\mathrm{PBE}}_{11001111000}\right)_{\varepsilon=0} \sim \int \frac{\mathrm{d}z}{\sqrt{P_4(z)}}$



- Elliptic curve

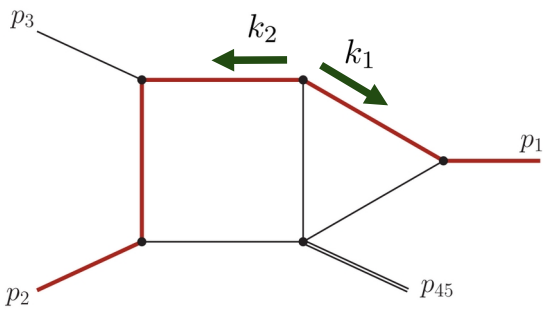
$$y^2 = P_4(z) = z(z-4m_t^2)(z-a_-)(z-a_+)$$



Elliptic sector $I^{\mathrm{PBE}}_{11001111000}$

- Basis of integrals (3 MIs)

$$\mathcal{J}^{(\mathrm{PBE})}_{39} = \varepsilon^4 d_{23} I^{(\mathrm{PBE})}_{11001111000}\,[2\,k_2\cdot p_1]$$
$$\mathcal{J}^{(\mathrm{PBE})}_{40} = \varepsilon^4 (1-2\varepsilon) d_{23} I^{(\mathrm{PBE})}_{11001111000}\,[2\,k_1\cdot p_3]$$
$$\mathcal{J}^{(\mathrm{PBE})}_{41} = \varepsilon^4 d_{23}(d_{12}+d_{23}-d_{45}+m_t^2)\, I^{(\mathrm{PBE})}_{11001111000}$$



- DEs are quadratic in ε

$$\mathrm{d}\,\mathcal{J} = \begin{pmatrix} *+*\varepsilon & * & *\varepsilon \\ (2\varepsilon-1)(*+*\varepsilon) & *+*\varepsilon & (2\varepsilon-1)*\varepsilon \\ *+*\varepsilon & * & *\varepsilon \end{pmatrix}\cdot\mathcal{J} + (\text{sub-sectors})$$

- Elliptic curve

$$y^2 = P_4(z) = z(z-4m_t^2)(z-a_-)(z-a_+)$$



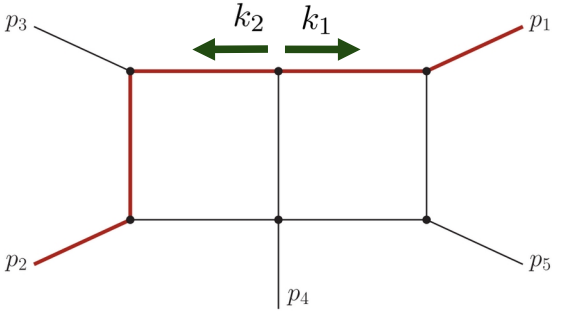
Nested square root sector $I^{(\mathrm{PBE})}_{11101111000}$

- Next-to Maximal Cut of the corner integral

$$I^{(\mathrm{PBE})}_{11101111000}\Big|^{\mathrm{MC}\setminus\{D_1\}}_{\varepsilon\to 0} \propto \int \frac{\mathrm{d}z_1\mathrm{d}z_4\mathrm{d}z_9}{z_1\sqrt{\mathcal{P}_1}\mathcal{P}_2\overline{\mathcal{P}}_2}$$

$\mathcal{P}_1$ quadratic polynomial

$\overline{\mathcal{P}}_2 = \mathcal{P}_2(r_1 \to -r_1)$



- 7 MIs

$$\mathcal{J}^{(\mathrm{PBE})}_5 = \frac{\varepsilon^4}{d_{45}} I^{(\mathrm{PBE})}_{1101111000}\left[\frac{\mathcal{P}_2-\overline{\mathcal{P}}_2}{2}\right]$$
$$\mathcal{J}^{(\mathrm{PBE})}_6 = \frac{\varepsilon^4}{d_{45}} I^{(\mathrm{PBE})}_{1101111000}\left[\frac{\mathcal{P}_2+\overline{\mathcal{P}}_2}{2}\right]$$

Following procedure of
[Becchetti,Canko,Chestnov,Peraro,Pozzoli,Zoia '25]

$$\mathcal{J}^{(\mathrm{PBE})}_7 = \frac{1}{\varepsilon}\frac{d_{45}m_t^2(d_{23}+d_{34})^2 - 2d_{13}d_{15}d_{23}d_{24}}{d_{15}r_2}\frac{\partial \mathcal{J}^{(\mathrm{PBE})}_{11}}{\partial d_{34}} + \dots$$

linear combination of derivatives of other MIs to achieve ε-linear DEs

$$\mathcal{J}^{(\mathrm{PBE})}_8 = \varepsilon^4 r_1 d_{15}\, I^{(\mathrm{PBE})}_{11101111000}\left[\frac{\mu_{12}}{(k_1-p_{15})^2}\right]$$
$$\mathcal{J}^{(\mathrm{PBE})}_9 = \varepsilon^4 r_1\, I^{(\mathrm{PBE})}_{11101111000}[\mu_{12}]$$
$$\mathcal{J}^{(\mathrm{PBE})}_{10} = \varepsilon^4 d_{15}d_{23}\, I^{(\mathrm{PBE})}_{11101111000}\left[(k_2+p_1)^2\right]$$
$$\mathcal{J}^{(\mathrm{PBE})}_{11} = \varepsilon^4 r_2 d_{15}\, I^{(\mathrm{PBE})}_{11101111000}$$

Numerators found requiring the integrands to have only simple poles

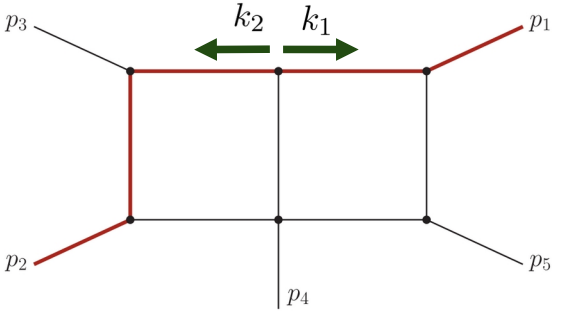
Nested square root sector $I^{(\mathrm{PBE})}_{11101111000}$

- Next-to Maximal Cut of the corner integral

$$I^{(\mathrm{PBE})}_{11101111000}\Big|^{\mathrm{MC}\setminus\{D_1\}}_{\varepsilon\to 0} \propto \int \frac{\mathrm{d}z_1\mathrm{d}z_4\mathrm{d}z_9}{z_1\sqrt{\mathcal{P}_1}\mathcal{P}_2\overline{\mathcal{P}}_2}$$

$\mathcal{P}_1$ quadratic polynomial

$\overline{\mathcal{P}}_2 = \mathcal{P}_2(r_1 \to -r_1)$



- DEs are linear in ε

$$\mathrm{d}\mathcal{J} = \left[\begin{pmatrix} X & Y & \mathbf{0} \\ Y & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{5\times 5} \end{pmatrix} + \mathcal{O}(\varepsilon)\right] \cdot \mathcal{J} + (\text{sub-sectors})$$

- DEs ε-factorised after the rotation

$$\begin{pmatrix} \mathcal{J}^{(\mathrm{PBE})}_5 \\ \mathcal{J}^{(\mathrm{PBE})}_6 \end{pmatrix} \mapsto \begin{pmatrix} n_2^+ & 0 \\ 0 & n_2^- \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{J}^{(\mathrm{PBE})}_5 \\ \mathcal{J}^{(\mathrm{PBE})}_6 \end{pmatrix}$$

$$n_2^{\pm} = \pm\sqrt{A_\pm \pm B_\pm \sqrt{G(p_1,p_2,p_3,p_4)}} \qquad A_\pm, B_\pm \text{ rational}$$

- In the subsectors it couples to both elliptic curves of PBE
- For the numerical evaluation we use the basis without nested square-roots

(Potentially over-complete) basis of special functions

$$\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) = \sum_{k=0}^{4} \varepsilon^k \, \vec{\mathcal{J}}^{(\mathrm{F},k)}(\vec{x}) + \mathcal{O}\left(\varepsilon^5\right)$$

O(ε^4) is enough to compute the two-loop amplitude up to O(ε^0)!

$\vec{\mathcal{J}}^{(\mathrm{F},k)}(\vec{x})$ are graded polynomials with rational coefficients of $\underbrace{1, f_i^{(\mathrm{F},1)}, f_i^{(\mathrm{F},2)}, f_i^{(\mathrm{F},3)}, f_i^{(\mathrm{F},4)}}, f_i^{(\mathrm{F},4*)}$

Iterated integrals of logarithmic one-forms, algebraic independent, algorithmic construction

[Gehrmann, Henn, Lo Presti, 2018]
[Chicherin, Sotnikov, 2020]
[Chicherin, Sotnikov, Zoia, 2022]
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2024]

Elliptic and nested square root functions, appear at O(ε^4) in the MIs and O(ε^0) in the amplitude, have not a well-defined trascendental weight, defined by coupled DEs, potentially over-complete basis

DEs for $f_i^{(\mathrm{F},k)}$:

- do not depend on ε
- have fewer one-forms (information at all order in ε truncated)

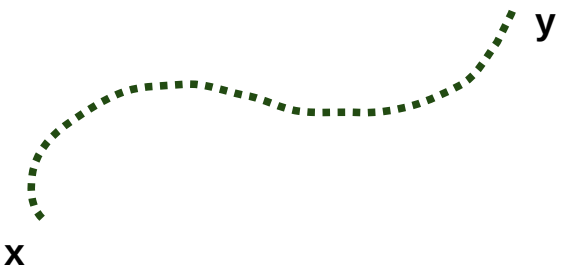
more efficient numerical evaluation!

family	MI	$f^{(\mathrm{F},1)}$	$f^{(\mathrm{F},2)}$	$f^{(\mathrm{F},3)}$	$f^{(\mathrm{F},4)}$	$f^{(\mathrm{F},4^*)}$	all $f^{(\mathrm{F},k)}$
PB_D	100	6	10	46	85	0	147
PB_E	154	6	10	53	110	13	192
PB_F	142	6	10	51	102	16	185

Numerical integration of DEs

- **Generalised series expansions** [Liu, Ma, Wang ‘17; Moriello ‘19]

  Automated codes
  - DiffExp [Hidding ‘20]
  - AMFlow [Liu, Ma ‘22]
  - SeaSyde [Armadillo, Bonciani, Devoto, Rana, Vicini ‘22]
  - LINE [Prisco, Ronca, Tramontano ‘25]



- **Chebyshev polynomial expansions**

  [Liu, Zhang ‘26]
  [Abreu, Guerreiro, Page ‘26]

- **Runge-Kutta**

  (e.g. ODEINT Boost Library)

- **Bulirsh-Stoer**

  (e.g. ODEINT Boost Library)

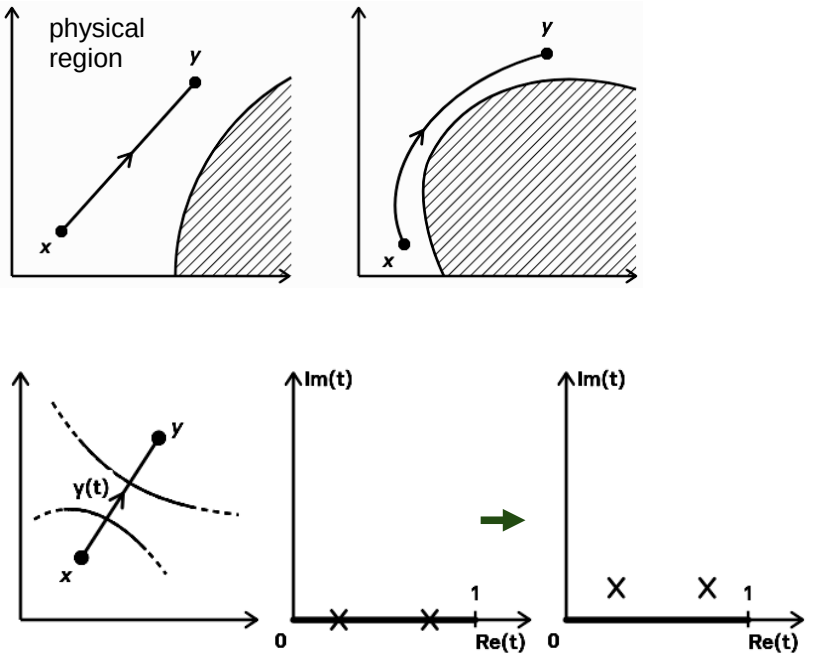
  [Czakon, Niggetiedt ‘20]
  [Czakon, Harlander, Klappert, Niggetiedt ‘21]
  [Rosas, Bobadilla ‘25]
  [Czakon et al. ‘25] → ttj (square roots)
  [Czakon, Tancredi ‘26]
  [Baur, Duhr ‘26]
  → arbitrary geometries

Numerical integration of DEs

- High-precision boundary values (70 digits) at a few random points in the physical region with AMFlow [Liu,Ma ‘22]
- Numerical solution of DEs with Bulirsh-Stoer algorithm implemented in Boost Library

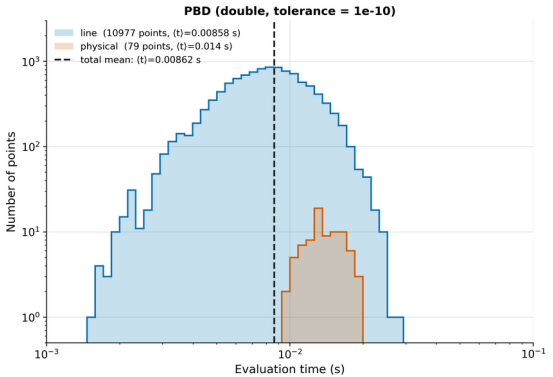
Main features of our integration:

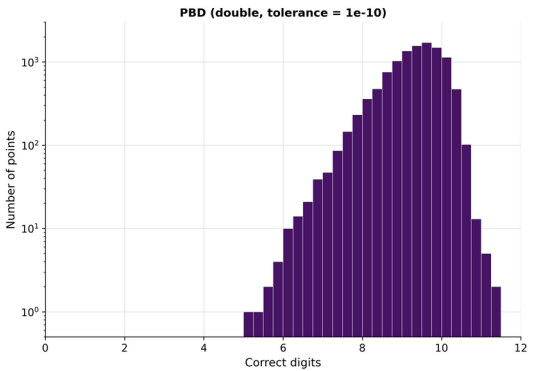
- The path never leaves the physical region, the straight line is preferred if possible
- Spurious singularities are avoided giving an imaginary part to the Mandelstam variables
- The accuracy is estimated by performing the integration from two different boundary points
- Derivative expressions are optimised with a ‘common subexpression elimination’ algorithm as implemented in FORM [Kuipers,Ueda,Vermaseren ‘13]



Numerical integration of DEs

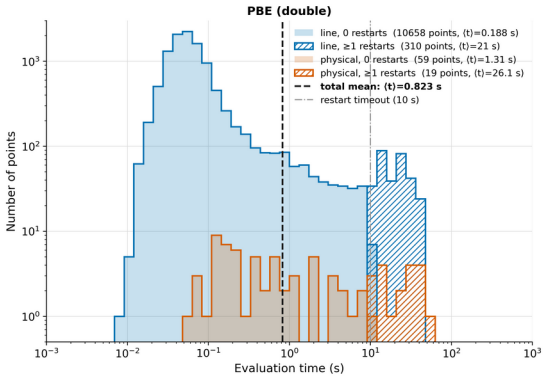
PBD

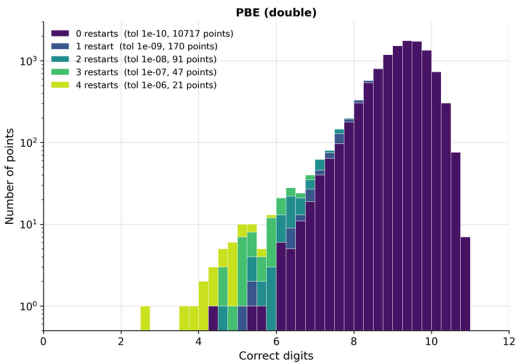




$\langle t \rangle = 0.01\,s$

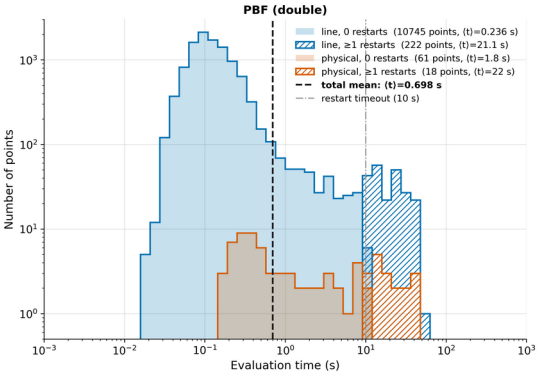
PBE

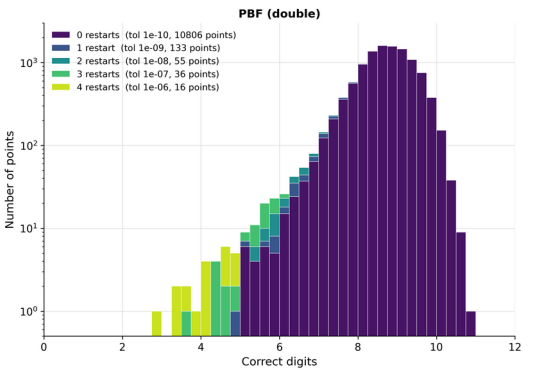




$\langle t \rangle = 0.82\,s$

PBF





$\langle t \rangle = 0.70\,s$

Conclusions

- We have built a basis of algebraically independent functions for the dlog part of the two-loop leading color amplitudes relevant for tt+photon production at hadron collider, the complicated (elliptic and nested square root) mathematical structure is encoded by few functions, potentially not independent, that appear only in the finite remainder
- We are reasonably satisfied with the numerical evaluation of the special functions, already suitable for a phenomenological application
- For the future a richer mathematical structure is waiting us...will this approach continue to be good enough for phenomenological applications? If so, can be automatised?
- Numerical evaluation of rational coefficients in the amplitude is often the bottleneck...can we choose bases of special functions which make them simpler?

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Thank you!