SOLVING DIFFERENTIAL EQUATION WITH SERIES EXPANSION TECHNIQUES

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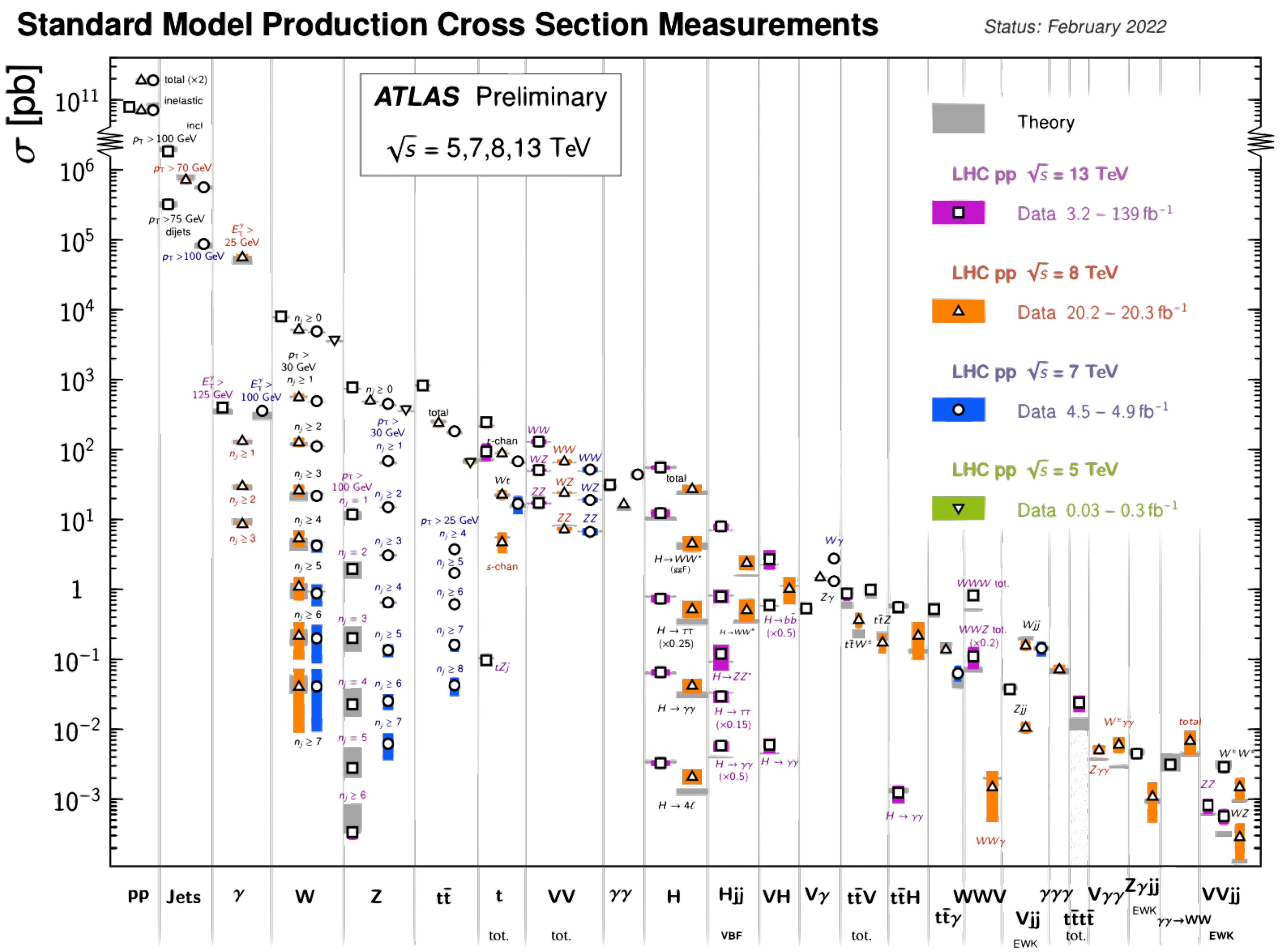
The Standard Model

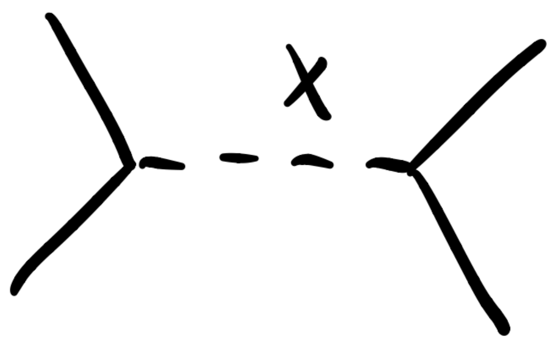
- The **Standard Model** is the best theory, so far, for describing the elementary particles and their interactions;

- During the years it has proven itself very successful in explaining and predicting with **extreme precisions** a big variety of phenomena in fundamental interactions, spanning several orders of magnitude;
- The discovery of the **Higgs boson** in 2012 confirmed one of the most important predictions of the Standard Model.

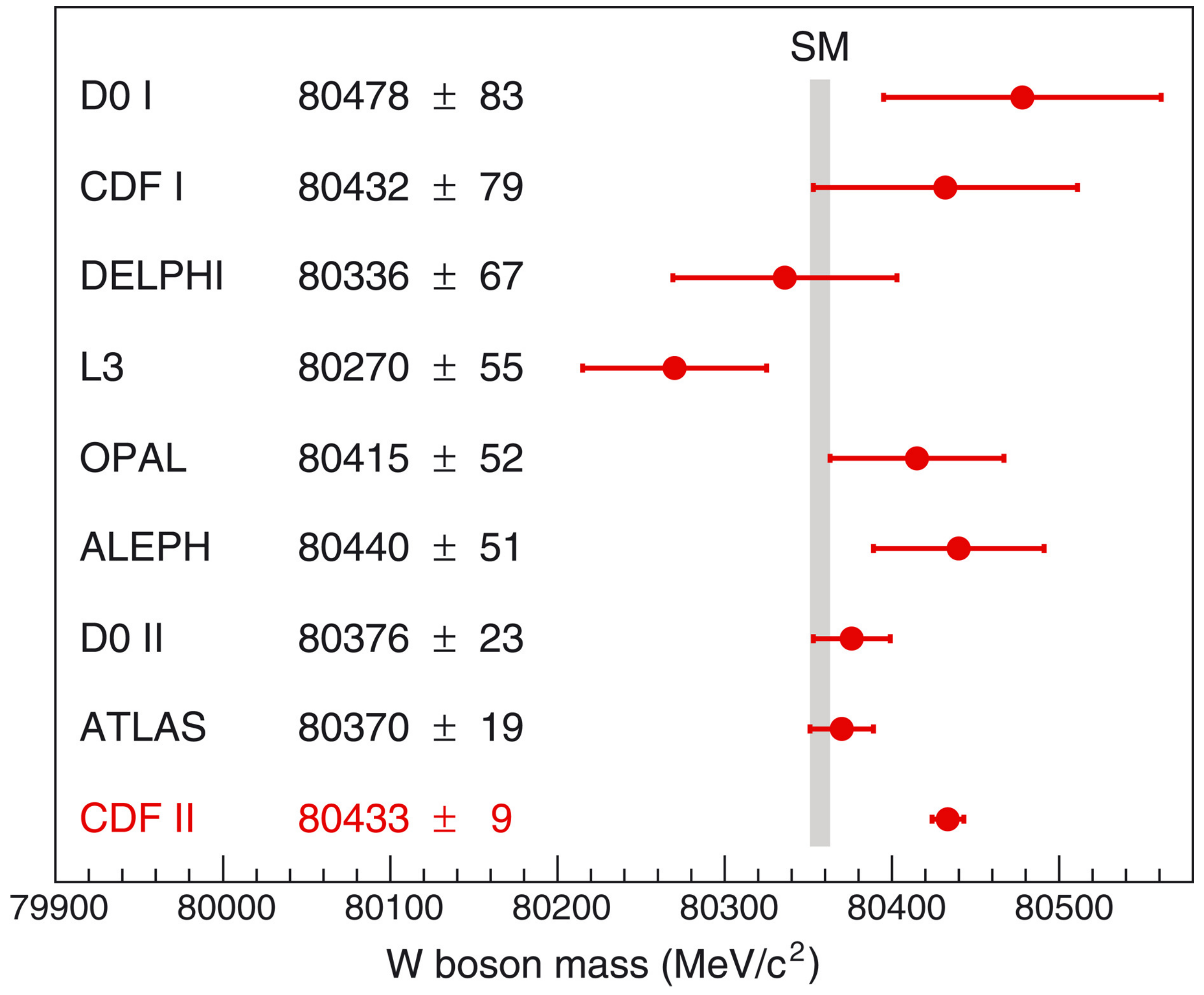


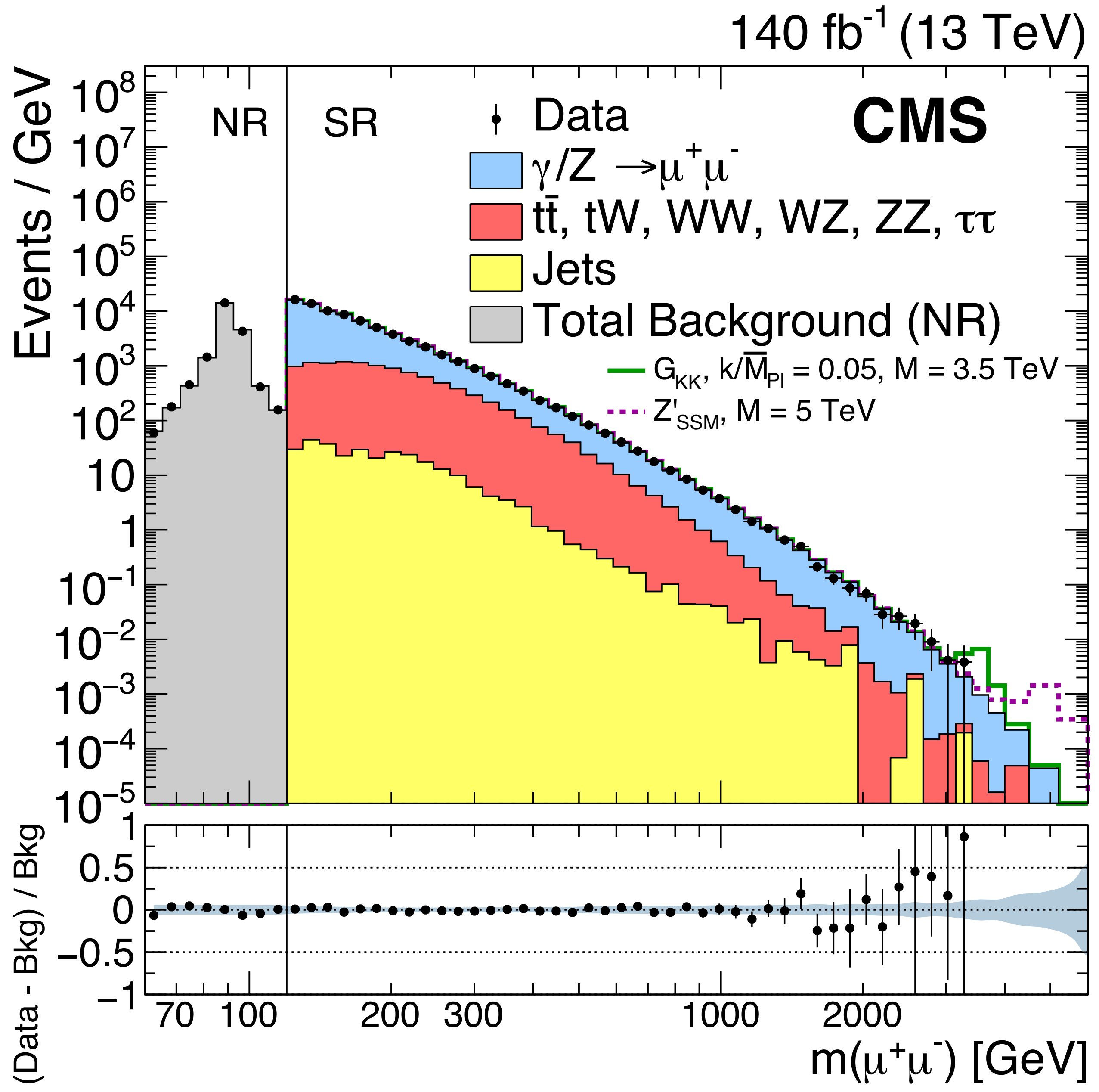
What’s next?

- The SM has still many open questions, e.g. **gravity**, **neutrino masses**, **dark matter**, …
- In order to answer those questions we need to find a model which goes **Beyond the Standard Model** (BSM). However, no experimental evidence of any BSM model has been found in the last years;
- New physics effects could still enter in virtual corrections, leading to some **deviations** of experimental measurements from theoretical predictions.



- In order to appreciate, and then interpret through a dedicated analysis, such differences, we need **theoretical predictions at least as accurate as the experimental value**.

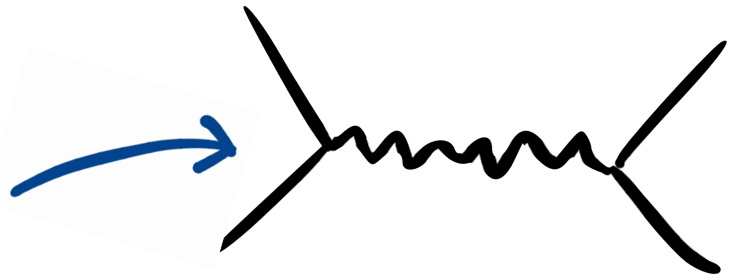




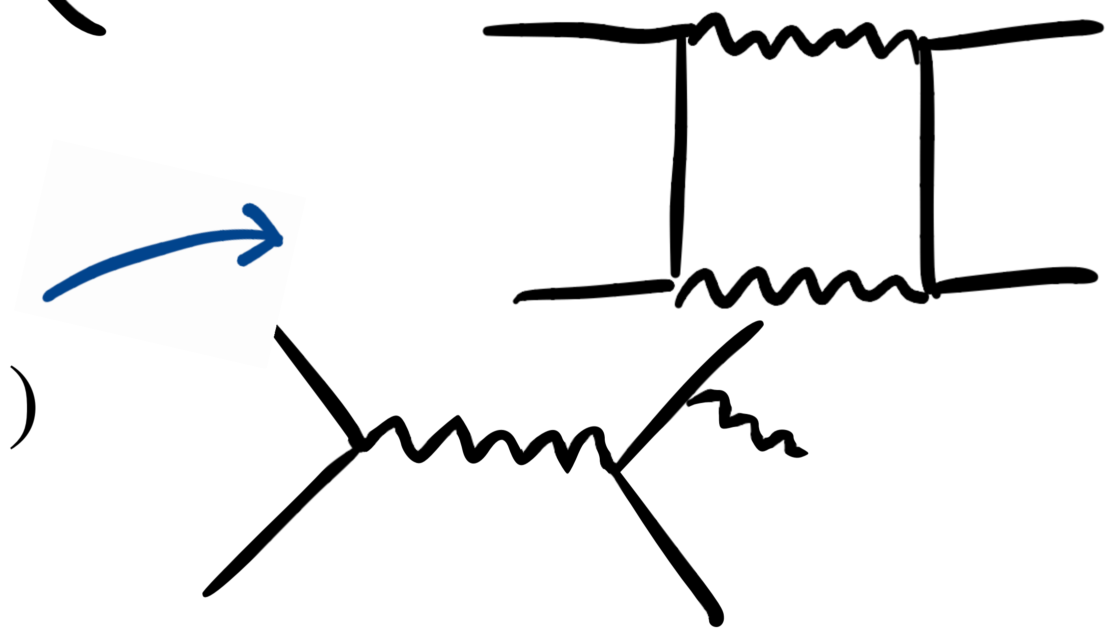
Higher order corrections

- In order to have more precise theoretical predictions we have to include **higher order corrections**, either in QCD or in EW;

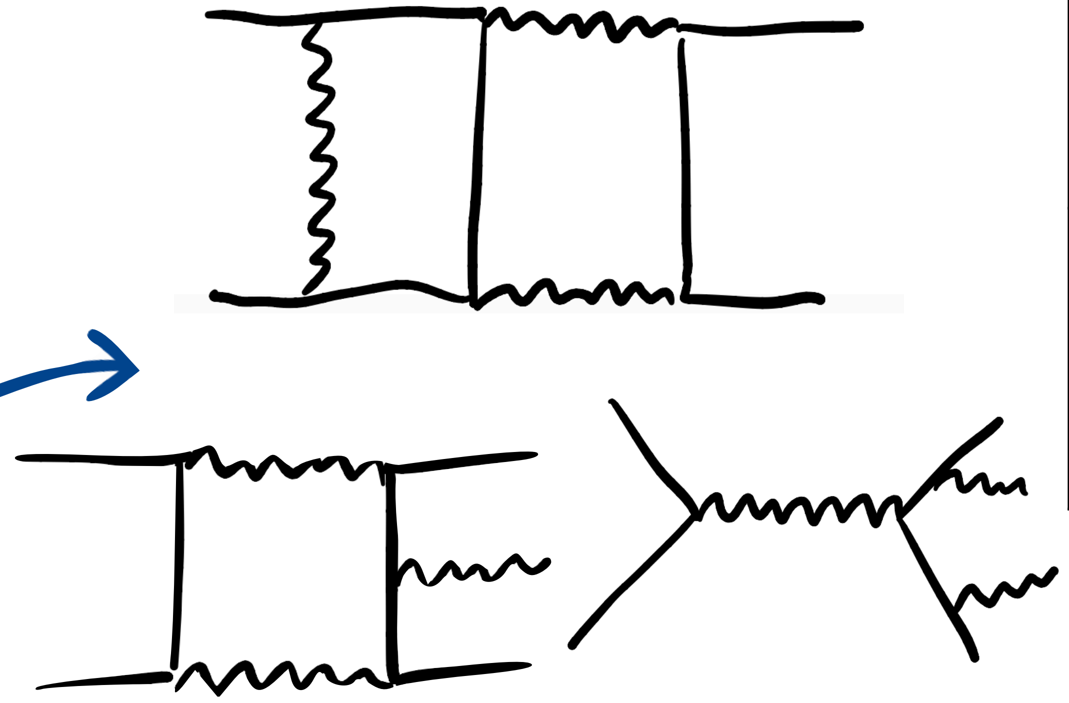
LO $\sigma^{(0,0)}$



NLO $+\alpha_S\, \sigma^{(1,0)} + \alpha\, \sigma^{(0,1)}$



NNLO $+\alpha_S^2\, \sigma^{(2,0)} + \alpha_S\, \alpha\, \sigma^{(1,1)} + \alpha^2\, \sigma^{(0,2)} + \dots$



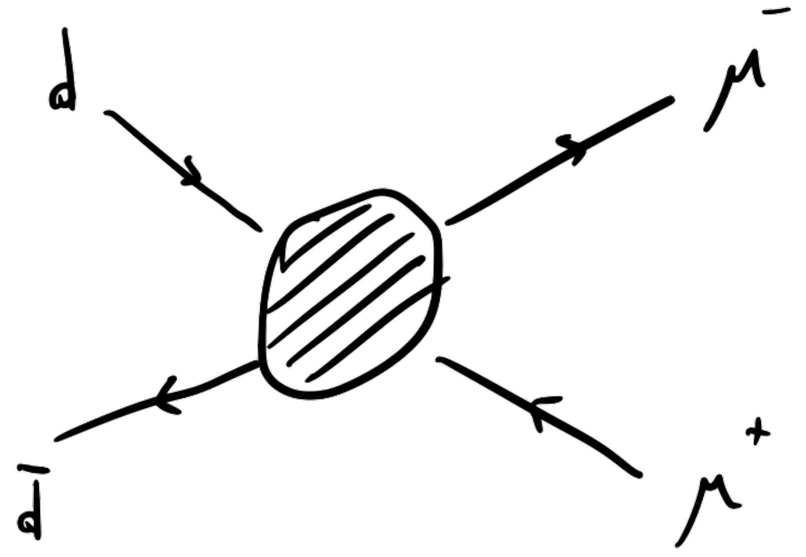
State of the Art	
$2 \to 1$	N3LO QCD NNLO Mixed NLO EW
$2 \to 2$	N3LO QCD NNLO Mixed NLO EW
$2 \to 3$	NNLO QCD NLO EW

- One of the main bottlenecks in these calculations comes from the evaluations of **virtual corrections** due to the **high number of Feynman integrals with different energy scales**;

Workflow of a Multi-loop computation

Process definition

- $\mathcal{O}(\alpha_S^2)$, $\mathcal{O}(\alpha_S \alpha)$, $\mathcal{O}(\alpha^2)$, …
- Which particles are massless?
- …

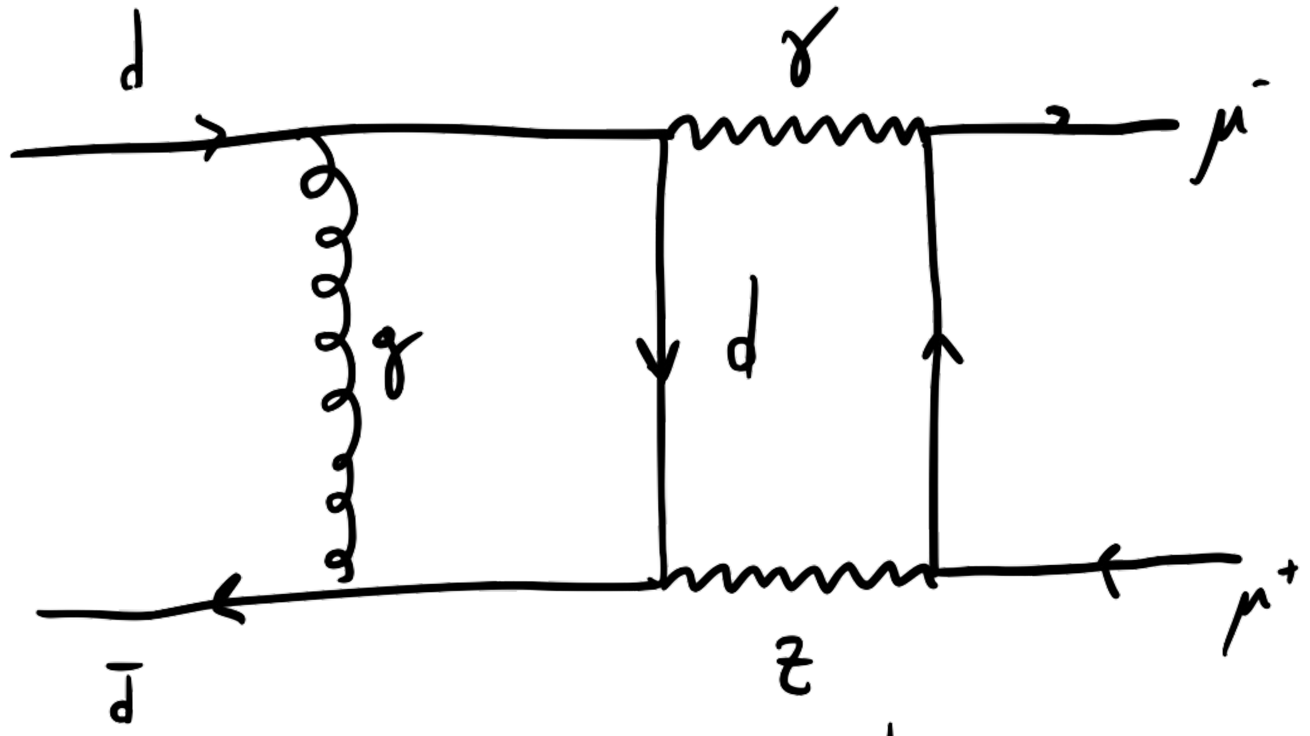


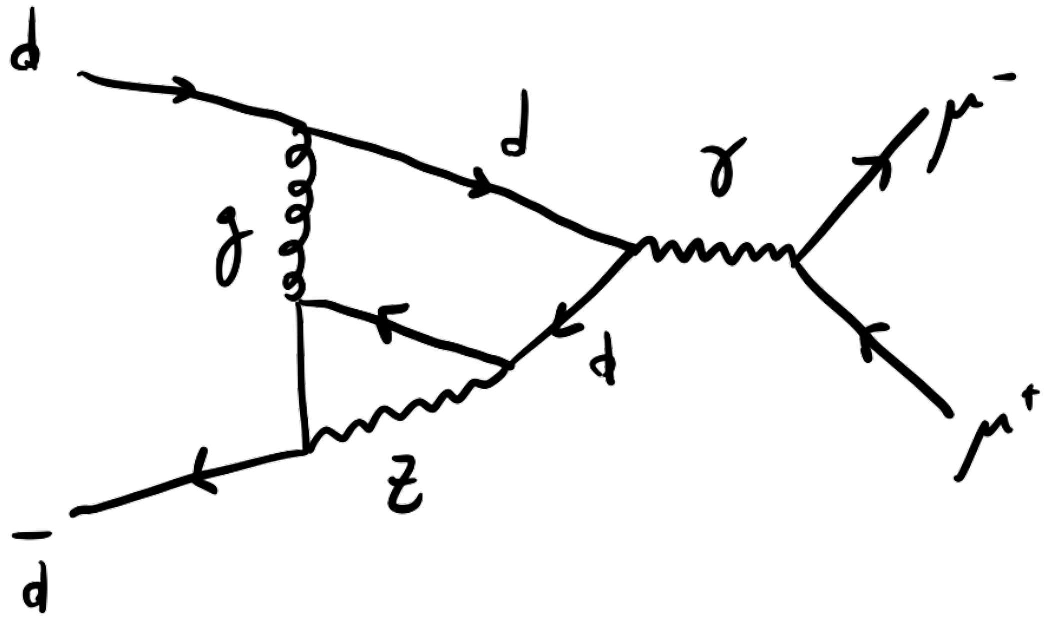
Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

- Some publicly available code: **FeynArts** or **QGRAF**





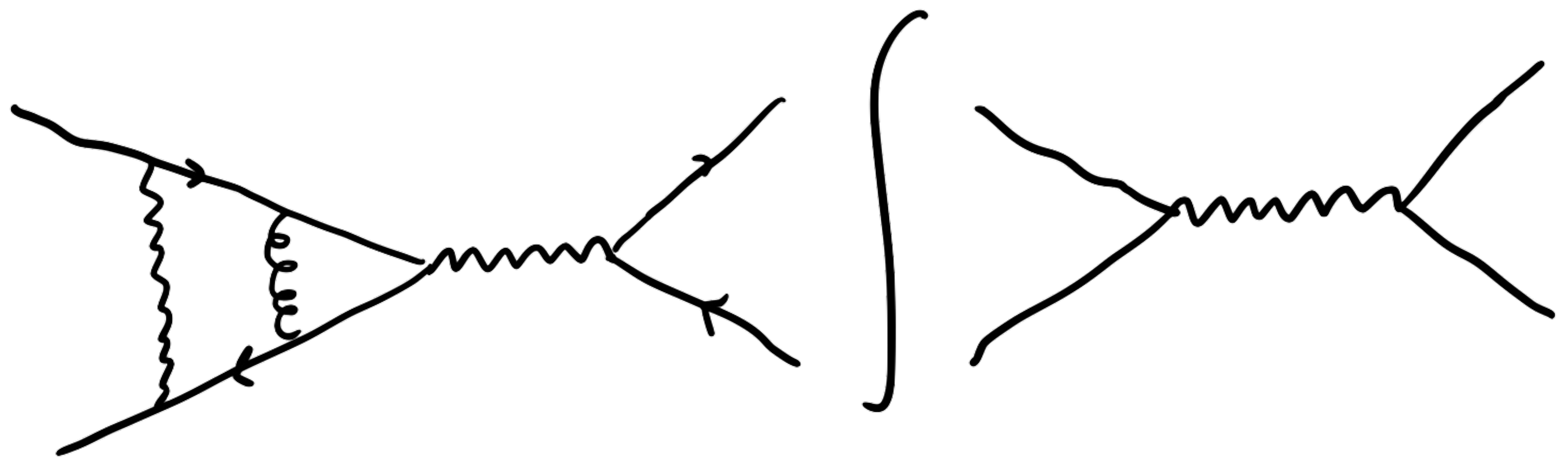
Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

Computation of interference terms

- Simplifying expressions;
- Handling γ^5 in d dimensions;
- Computing traces of γ-matrices;
- Reduce tensor loop-integrals to scalar ones;



Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

Computation of interference terms

Reduce to a set of Master Integrals

- Publicly available code: **KIRA**, **FIRE** or **REDUZE2**, …

$$\sum_{i=1}^{N} c_i I_i \qquad \longrightarrow \qquad \sum_{i=1}^{m} \tilde{c}_i \, MI_i \,, \quad m \ll N$$

Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

Computation of interference terms

Reduce to a set of Master Integrals

Evaluation of MIs

- Different approaches are possible: Feynman parameters, Monte Carlo integration, Tropical Geometry, …
- One possibility is the Method of **differential equations** (with a semi-analytical approach);
- **Complex masses** for gauge bosons;

Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

Computation of interference terms

Reduce to a set of Master Integrals

Evaluation of MIs

Subtraction of divergences

- Counter-terms for **UV renormalisation**;
- Subtraction of **IR divergences**;

Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

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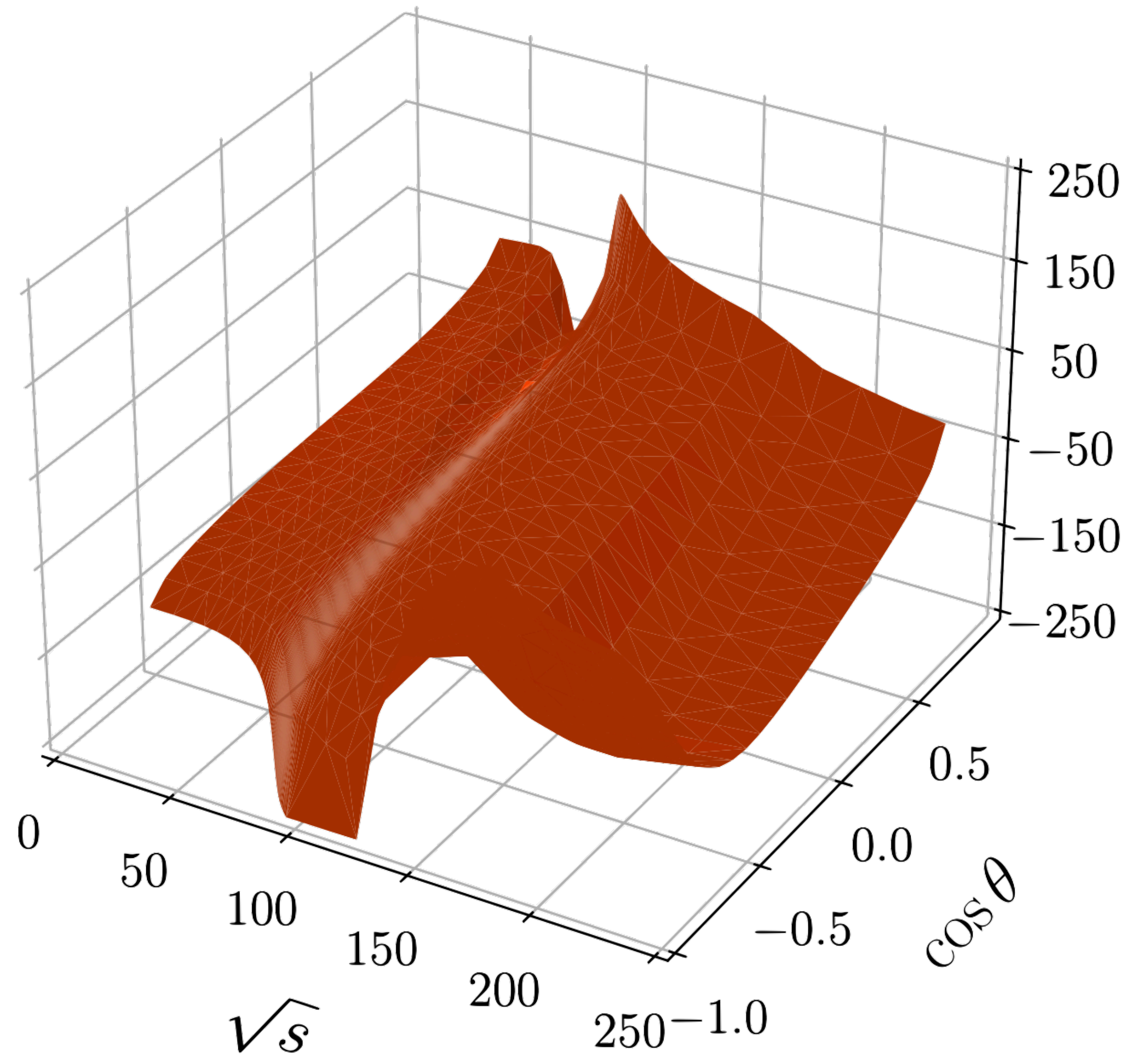
Reduce to a set of Master Integrals

Evaluation of MIs

Subtraction of divergences

Production of a numerical grid



- Create a **numerical grid**;
- Combine with real contributions and perform a **Monte-Carlo** integration over phase-space;
- **Phenomenology**

Workflow of a Multi-loop computation

Process definition

Generation of Feynman diagrams

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Reduce to a set of Master Integrals

Evaluation of MIs

Subtraction of divergences

Production of a numerical grid

THIS TALK!

Evaluating Feynman integrals

- What we would like to compute are objects like this:

$d = 4 - 2\epsilon$

$$I(\alpha_i; s_j, d) = \int \prod_{k=1}^{l} \frac{d^d q_k}{i\pi^{d/2}} \frac{1}{\mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n}}$$

kinematic variables

$e.g.\ \ (p_1 - q_1)^2 - m^2 + i\delta$

- A given set of denominators $\mathcal{D}_i$ constitutes an **integral family**. Inside an integral family an integral is uniquely identified by the set of the different powers α_i to which the denominators are raised.
- Using **Integration by Parts** (IBP) identities, we can express all the integrals of the given integral family in terms a smaller subset, the so-called **Master Integrals**.

How to compute the Master Integrals?

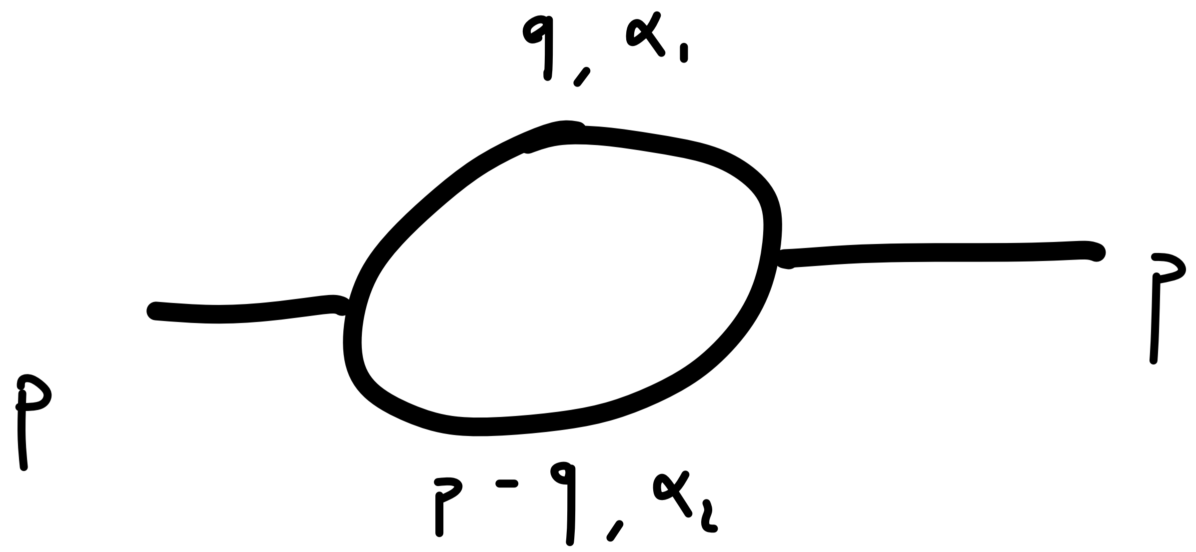
- **Many techniques** have been developed during the years, each with pros and cons. Here I will focus on the **method of differential equations**.

- The idea is that by differentiating a master w.r.t. a kinematical invariants we obtain a **first order linear differential equations**, whose solution is the master integrals we are interested in.

$$\frac{\partial}{\partial s_k} I(\alpha_i; s_j, d) = \sum \textit{scalar integrals} \underset{IBP}{=} \sum \textit{master integrals}$$

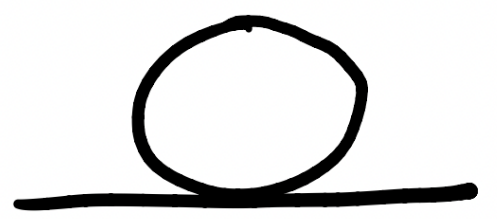
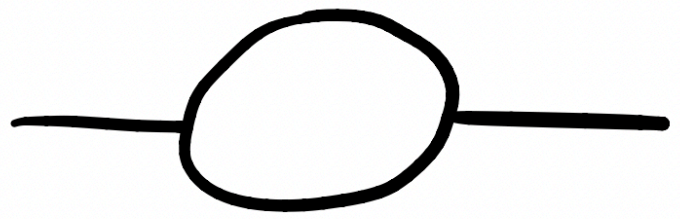
- By repeating the same process for every master integral we obtain a **system of first order linear and homogeneous differential equations**.

A very simple example: 1L bubble

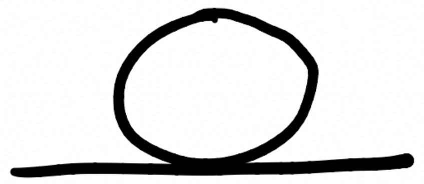


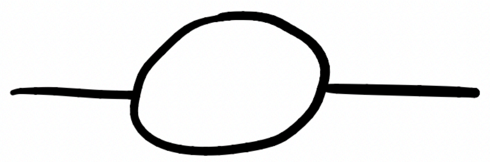
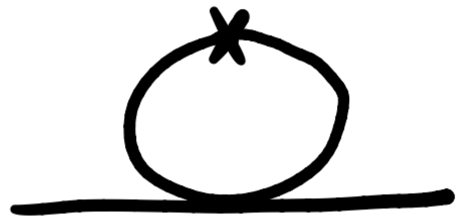
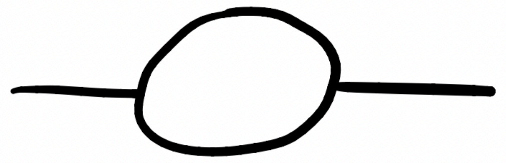
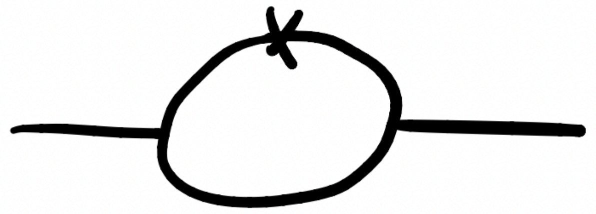
$\Longleftrightarrow$ $I_{\alpha_1\,\alpha_2}(p^2, m^2, d) = \int \frac{d^dq}{i\pi^{d/2}} \frac{1}{\left[q^2 - m^2\right]^{\alpha_1}\,\left[(q-p)^2 - m^2\right]^{\alpha_2}}$

- This problem has **2 kinematic invariants**, p^2 and m^2, and **2 master integrals**: I_{10} and I_{11}

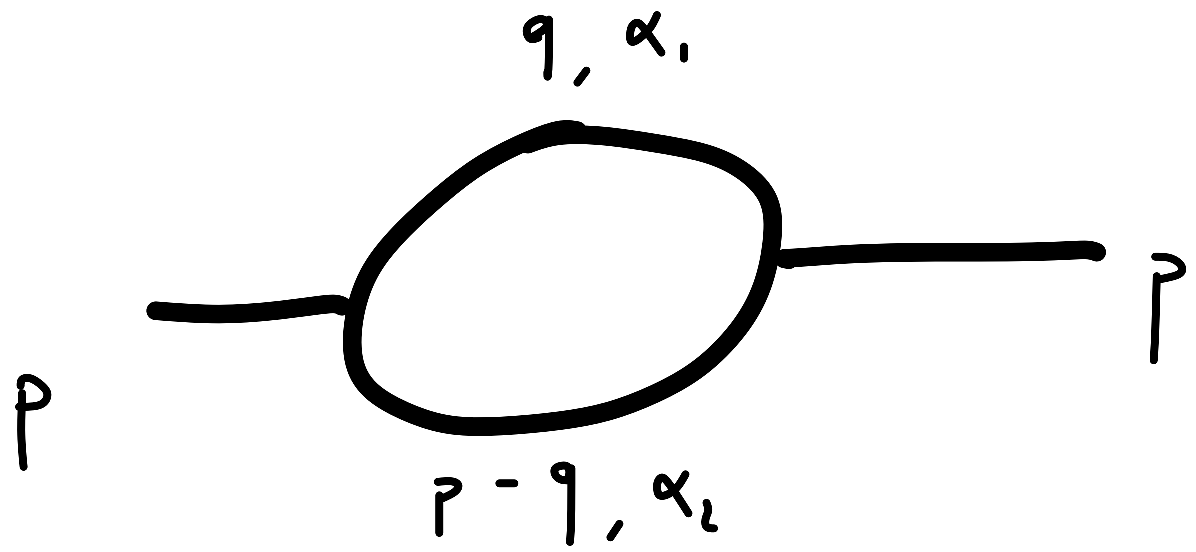
$I_{10} =$  $I_{11} =$ 

- By differentiating w.r.t. p^2 we obtain

$\frac{d}{dp^2}$  $= 0$

$\frac{d}{dp^2}$  $= \frac{1}{2p^2}$  $- \frac{1}{2p^2}$  $- \frac{1}{2}$ 

A very simple example: 1L bubble



$$\Longleftrightarrow \quad I_{\alpha_1\,\alpha_2}(p^2,m^2,d) = \int \frac{d^dq}{i\pi^{d/2}}\frac{1}{\left[q^2-m^2\right]^{\alpha_1}\left[(q-p)^2-m^2\right]^{\alpha_2}}$$

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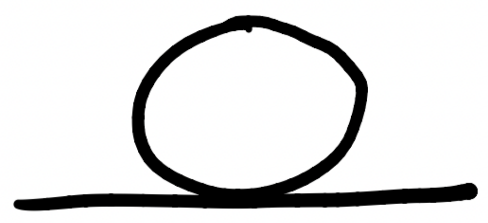
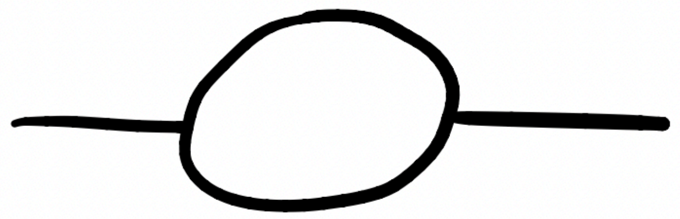
$I_{10} =$  $\qquad I_{11} =$ 

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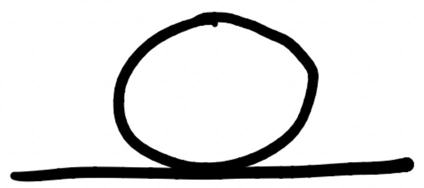
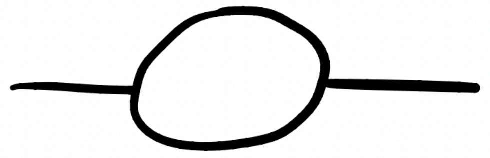
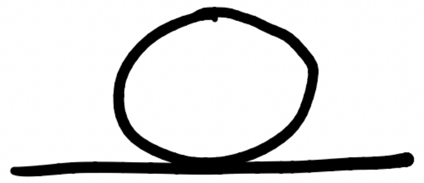
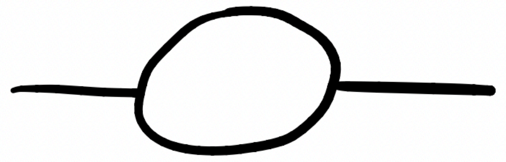
$\frac{d}{dp^2}$  $= 0$

$\frac{d}{dp^2}$  $= \frac{d-2}{p^2(4m^2-p^2)}$  $- \frac{(d-4)p^2+4m^2}{2p^2(4m^2-p^2)}$ 

What are we looking for?

- So we just have to solve a system of first order differential equations… HOW?
- Ideally, we would like:
  - A method **easy to automatise**

```
ImportMasterIntegrals["my_master_integrals"]
ABISS: Succesfully imported 380 master integrals
```

  - A solution **compact and easy to handle** to allow for simplifications

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{\log^2(-x)}{2}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\,(1-x) = \frac{\pi^2}{6} + \log(x)\log(1-x)$$

  - A solution **fast to evaluate** to be implemented in a Monte-Carlo
  - To have **high control on numerical precision**

$$\mathcal{O}(10-10^2) \quad 2\mathrm{Re}\mathcal{M}^{(2)}\mathcal{M}^{(0)*} = \sum_i c_i\, MI_i \quad \mathcal{O}(10^{-10}-10^{10})$$

Analytical solution

- There are many possibilities to solve the system, each with pros and cons
- The first method is to solve it **analytically**. This is, by far, the preferable method.

$$I_{1,1}^{(finite)}(p^2, m^2) = 2 - \gamma_E - \log m^2 + \frac{m^2}{p^2}\left(\frac{1}{r} - r\right)\log r \qquad \textit{with} \qquad r = \frac{-p^2 + 2m^2 + \sqrt{(p^2 - 2m^2)^2 - 4m^4}}{2m^2}$$

- The result is provided in closed form as a combination of elementary and special functions, such as **Generalised PolyLogarithms**. Of these functions we know functional relations and series expansion;

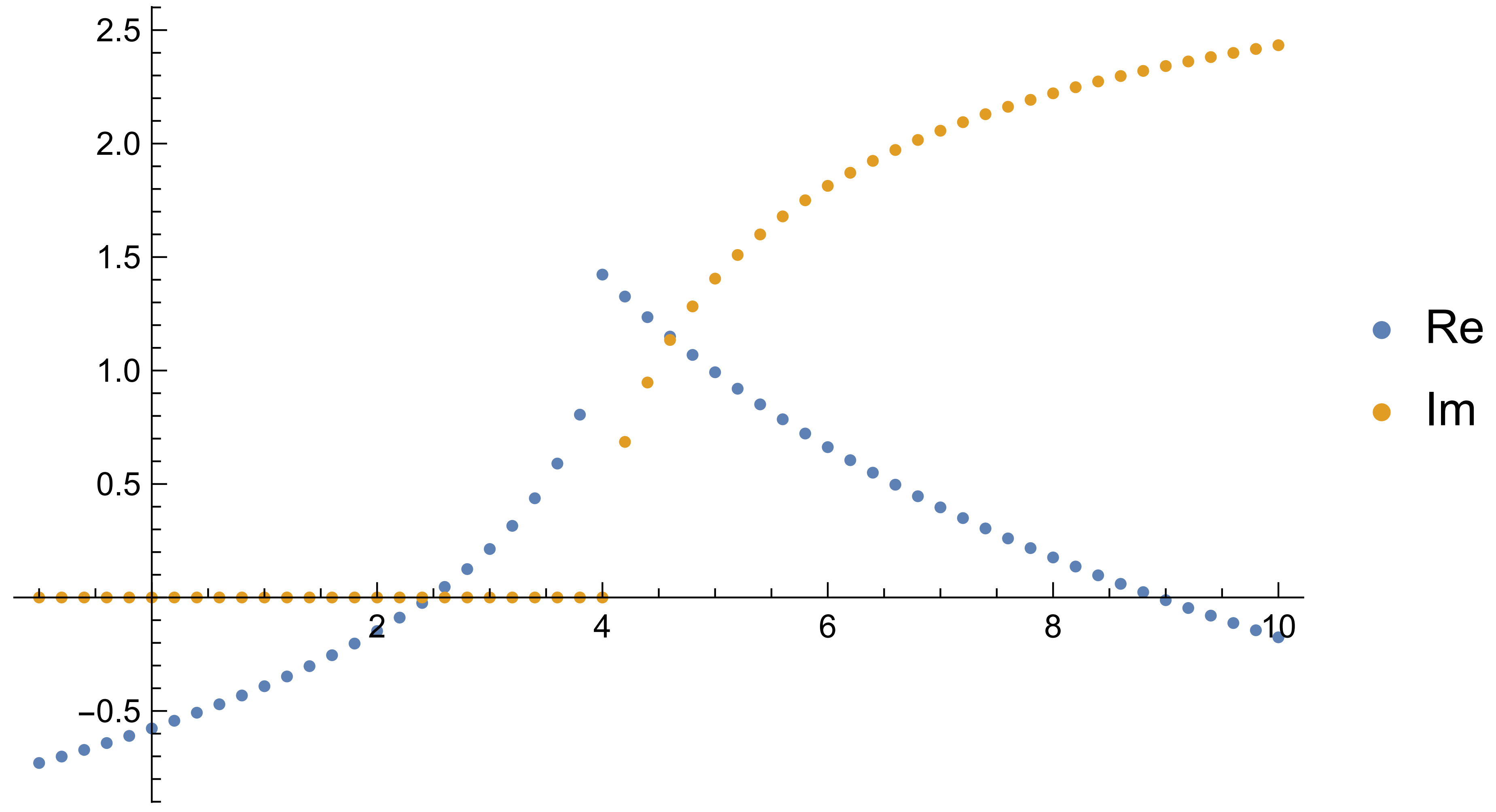
$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; z) \qquad \textit{and} \qquad G(\vec{0}_n; z) = \frac{1}{n!}\log^n z$$

- However, especially when increasing the number of scales or legs, an analytical expression in terms of known classes of functions might not be available. Moreover, the numerical evaluation of the result might require a long time with external libraries.

Numerical solution

- The second possibility is to solve the equations **numerically**. Now the result is provided as a **numerical grid**.

$$I_{1,1}^{(finite)}(p^2, m^2 = 1) \quad :$$



- This can be done with methods such as Runge-Kutta. There are some examples in literature, however this has not received too much attention. The main problem is the difficulty in controlling the numerical precision of the solution.

Semi-analytical solution

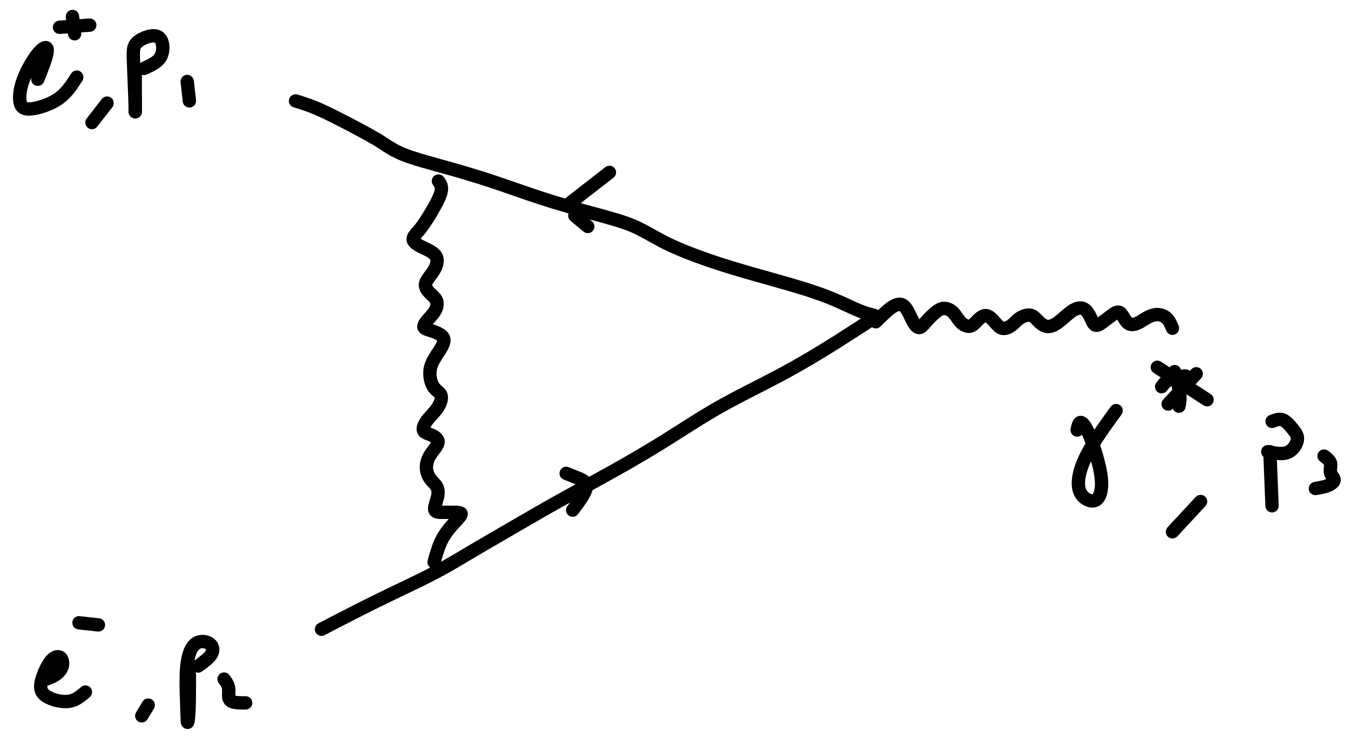
- A third possibility could be to use a **semi-analytical approach**. In this case the result is provided as a power series which can be easily evaluated in every point of the domain.

$$I_{1,1}^{(finite)}(p^2, m^2 = 1) = -\gamma_E + \frac{1}{6}p^2 + \frac{1}{60}(p^2)^2 + \frac{1}{420}(p^2)^3 + \frac{1}{2520}(p^2)^4 + \frac{1}{13860}(p^2)^5 + \dots$$

- The method has been firstly implemented in the Mathematica package **DiffExp** for a **real kinematic variable** *[F.Moriello, arXiv:1907.13234], [M.Hidding, arXiv:2006.05510]*
- The main advantage is that all the calculations can be carried out analytically.
- This method is quite **easy to automatise**. Provided that we have infinite time and space, we could achieve **arbitrary precision**. Moreover, once we have the solution, it can be evaluated numerically in a negligible amount of time.
- However, **series have a limited radius of convergence**, hence, an algorithm for performing the analytic continuation of the solution must be provided.

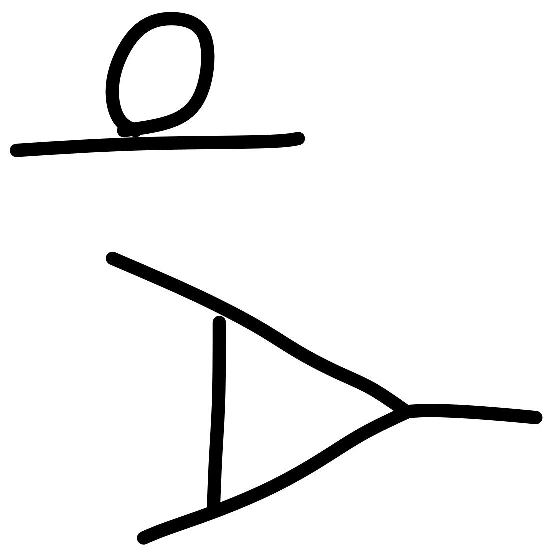
1L-QED vertex

$$p_1^2 = p_2^2 = m_e^2$$
$$(p_1 + p_2)^2 = s$$



$$I_{n_1 n_2 n_3} = \int \frac{d^d q}{i\pi^{d/2}} \frac{1}{\left[q^2\right]^{n_1} \left[(q+p_1)^2 - m_e^2\right]^{n_2} \left[(q-p_2)^2 - m_e^2\right]^{n_3}}$$

- This problem has **2 masters**, which can be chosen as the massive tadpole and the scalar triangle;



$$\frac{d}{ds} B_1 = 0$$

$$\frac{d}{ds} B_2 = \frac{\epsilon - 1}{s\, m_e^2\, (s - 4m_e^2)} B_1 + \frac{2m_e^2 - s - s\epsilon}{s\,(s - 4m_e^2)} B_2$$

- The singularities of the problem can be read from the coefficient matrix.

1L-QED vertex

- For simplicity we introduce an **adimensional variable** $x = s/m_e^2$. This reduces the number of parameters the problem depends on, thus speeding up the computation.
- The first step is to **separate each order in** ϵ. To do so we can read from the boundary conditions the minimum order in ϵ, and write $B_i = \sum_{j=\epsilon_{min}}^{+\infty} B_i^{(j)}\, \epsilon^j$. Then we can collect order by order in ϵ:

$$\mathcal{O}(1/\epsilon)\ :\quad \begin{aligned} &\frac{d}{dx}B_1^{(-1)} = 0 \\ &\frac{d}{dx}B_2^{(-1)} = -\frac{1}{x\,(x-4)}B_1^{(-1)} - \frac{x-2}{x\,(x-4)}B_2^{(-1)} \end{aligned}$$

$$\mathcal{O}(\epsilon^0)\ :\quad \begin{aligned} &\frac{d}{dx}B_1^{(0)} = 0 \\ &\frac{d}{dx}B_2^{(0)} = \frac{1}{x\,(x-4)}B_1^{(-1)} - \frac{1}{x\,(x-4)}B_1^{(0)} - \frac{1}{x-4}B_2^{(-1)} - \frac{x-2}{x\,(x-4)}B_2^{(0)} \end{aligned}$$

1L-QED vertex

- Let us start from $\mathcal{O}(1/\epsilon)$ and expand around $x = 0$. The first equation is trivial and gives: $B_1^{(-1)}(x) = 1$

$$\frac{d}{dx}B_2^{(-1)}(x) = -\frac{x-2}{x\,(x-4)}B_2^{(-1)}(x) - \frac{1}{x\,(x-4)}$$

$$B_2^{(-1)}(0) = \frac{1}{2}$$

- First of all we start from the **homogeneous equation**. For that we use the **Frobenius method**, i.e. we use the ansatz $B_2^{(-1),hom}(x) = x^r \sum_{i=0}^{\infty} c_i x^i$, with $r \in \mathbb{Q}$.

$$x^r\left[c_1 + 2xc_2 + 3x^2c_3 + \mathcal{O}(x^3)\right] + rx^{-1+r}\left[c_0 + xc_1 + x^2c_2 + \mathcal{O}(x^3)\right] =$$

$$-\frac{x-2}{x\,(x-4)}$$

$$B_2^{(-1)}$$

$$\frac{d}{dx}B_2^{(-1)}$$

$$= \left[\frac{1}{8} - \frac{1}{2x} + \frac{x}{32} + \frac{x^2}{128} + \mathcal{O}(x^3)\right] x^r\left[c_0 + xc_1 + x^2c_2 + \mathcal{O}(x^3)\right]$$

1L-QED vertex

- Let us expand everything

$$r\, c_0\, x^{-1+r} + (1+r)\, c_1\, x^r + (2+r)\, c_2\, x^{1+r} + (3+r)\, c_3\, x^{2+r} + \mathcal{O}(x^{3+r}) =$$

$$= -\frac{c_0}{2}\, x^{-1+r} + \left(\frac{c_0}{8} - \frac{c_1}{2}\right)\, x^r + \frac{1}{32}\left(c_0 + 4c_1 - 16c_2\right)\, x^{1+r} + \frac{1}{128}\left(c_0 + 4c_1 + 16c_2 - 64c_3\right) x^{2+r} + \mathcal{O}(x^{3+r})$$

- And collect the different powers of x:

$$\begin{aligned} r\, c_0 &= -\frac{1}{2} c_0 \\ (1+r)\, c_1 &= \frac{1}{8} - \frac{c_1}{2} \\ (2+r)\, c_2 &= \frac{1}{32}\left(1 + 4c_1 - 16c_2\right) \\ (3+r)\, c_3 &= \frac{1}{128}\left(1 + 4c_1 + 16c_2 - 64c_3\right) \end{aligned} \quad \longrightarrow \quad \begin{aligned} r &= -\frac{1}{2} \\ c_1 &= \frac{1}{8} c_0 \\ c_2 &= \frac{3}{128} c_0 \\ c_3 &= \frac{5}{1024} c_0 \end{aligned}$$

Variation of parameters

- Now that we have a solution to the homogeneous equation we can obtain a particular one with the method of **variation of parameters**, i.e. we look for a particular solution of the form: $B_2^{(-1),part}(x) = C(x)\, B_2^{(-1),hom}(x)$. If we substitute in the original equation we get

$$C'(x)\, B_2^{(-1),hom}(x) + \cancel{C(x)\, B_2^{(-1),hom'}(x)} = -\frac{x-2}{x\,(x-4)}\, \cancel{C(x)\, B_2^{(-1),hom}(x)} - \frac{1}{x\,(x-4)}$$

$$C'(x)\, B_2^{(-1),hom}(x) = -\frac{1}{x\,(x-4)} \quad \longrightarrow \quad C'(x) = -\frac{1}{x\,(x-4)} \left(B_2^{(-1),hom}(x)\right)^{-1}$$

$$C(x) = \int_0^x -\frac{1}{x'\,(x'-4)} \left(B_2^{(-1),hom}(x')\right)^{-1} dx'$$

$$\longrightarrow \quad B_2^{(-1),part}(x) = B_2^{(-1),hom}(x) \int_0^x -\frac{1}{x'\,(x'-4)} \left(B_2^{(-1),hom}(x')\right)^{-1} dx'$$

1L-QED vertex

- Now we can expand everything and integrate. Note that since we are expanding in x, **all the integrals are trivial**.

$$\left(B_2^{(-1),hom}(x)\right)^{-1}$$

$$-\frac{1}{x'\,(x'-4)}$$

$$B_2^{(-1),hom}(x)$$

$$B_2^{(-1),part}(x) = c_0\, x^{-1/2}\left(1+\frac{1}{8}x+\frac{3}{128}x^2+\frac{5}{1024}x^3+\mathcal{O}(x^4)\right)\times$$
$$\times\int_0^x\left[\frac{1}{4x'}+\frac{1}{16}+\frac{x'}{64}+\frac{x'^2}{256}+\frac{x'^3}{1024}+\mathcal{O}(x'^4)\right]\left[c_0^{-1}\,x'^{1/2}\left(-\frac{x'}{8}-\frac{x'^2}{128}-\frac{x'^3}{1024}+\mathcal{O}(x'^4)\right)\right]dx' =$$
$$=\frac{1}{2}+\frac{x}{12}+\frac{x^2}{60}+\frac{x^3}{280}+\mathcal{O}(x^4)$$

- The **complete solution** is obtained by combining the homogeneous one with the particular one.

$$B_2^{(-1)}(x) = c\,B_2^{(-1),hom}(x)+B_2^{(-1),part}(x) = c\,x^{-1/2}\left(1+\frac{x}{8}+\frac{3x^2}{128}+\frac{5x^3}{1024}+\mathcal{O}(x^4)\right)+\left(\frac{1}{2}+\frac{x}{12}+\frac{x^2}{60}+\frac{x^3}{280}+\mathcal{O}(x^4)\right)$$

1L-QED vertex

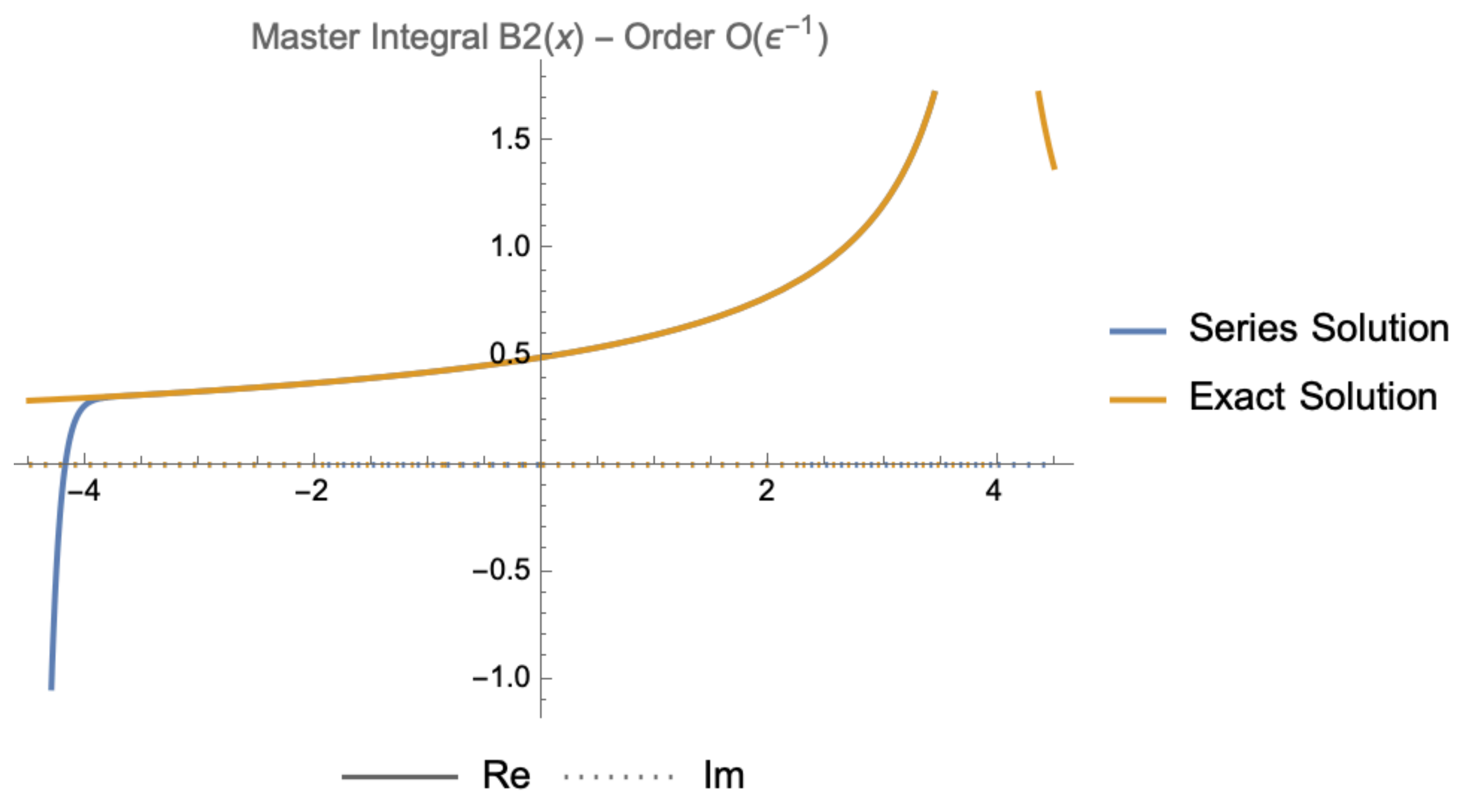
- The last thing to do is to fix the constant using the **boundary condition**: $B_2^{(-1)}(0) = \dfrac{1}{2}$

$$B_2^{(-1)}(x) = c\, x^{-1/2}\left(1 + \frac{x}{8} + \frac{3x^2}{128} + \frac{5x^3}{1024} + \mathcal{O}(x^4)\right) + \left(\frac{1}{2} + \frac{x}{12} + \frac{x^2}{60} + \frac{x^3}{280} + \mathcal{O}(x^4)\right) =$$

$$= \frac{1}{2} + \frac{x}{12} + \frac{x^2}{60} + \frac{x^3}{280} + \mathcal{O}(x^4)$$

$c = 0$

- From the differential equations we read the position of the singularities: $x = 0$ and $x = 4$. Since the series is centred in $x = 0$, this translate to the fact that the solution converges inside the interval $(-4,4)$;
- We will come back later on the analytic continuation.



Logarithmic expansion

- In the previous case we expanded on $x = 0$, which was a possibly singular point, however, the solution was regular. That means that $x = 0$ is a **pseudo-threshold**.
- We could have expanded around a **regular point**. In this case we always have $r \geq 0$, hence, the solution is a simple **Taylor series**.
- Another possibility is to expand on top of a **threshold**, e.g. in the 1L-QED vertex, $x = 4$. The solution could contain terms like:

$$\frac{1}{x-4} \qquad \text{or} \qquad \log(x-4)$$

- These terms could arise from variation of parameters method. In particular:

$$f^{part}(x) = f^{hom}(x) \int_0^x g^{non\ hom}(x') \ \left(f^{hom}(x')\right)^{-1} dx'$$

could contain either $1/x^m$ with $m > 1$ or $1/x$. At higher order in ϵ, $g^{non\ hom}(x)$ may directly contains $\log$.

Logarithmic expansion

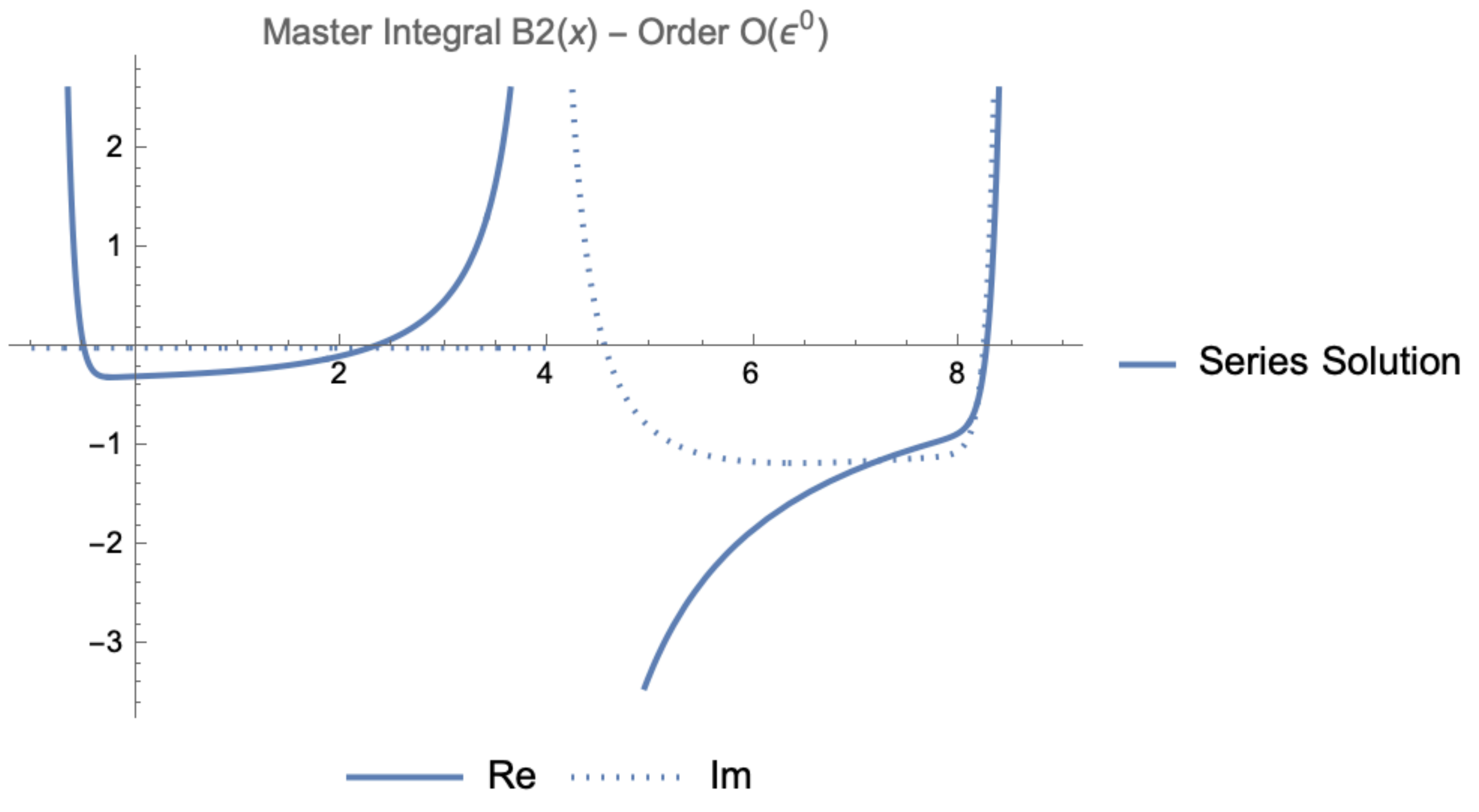
- For example, if we try to solve the order $\mathcal{O}(\epsilon^0)$ of the same problem, but this time around $x = 4$, we get:

$$B_2^{(0)}(x) = 1.28861 \; - 0.18699\,(x-4) \; + 0.0357314\,(x-4)^2 \; - 0.00748665\,(x-4)^3 \; + \mathcal{O}(x-4)^4 +$$

$$+\frac{1}{\sqrt{x-4}}\Big(-(4.9348 + 0.906688i) \; + (0.61685 + 0.113336i)\,(x-4) +$$

$$+(0.0240957 + 0.00442719i)\,(x-4)^3 \; + \mathcal{O}(x-4)^4\Big) +$$

$$+\frac{\log(x-4)}{\sqrt{x-4}}\Big(-1.5708i \; + 0.19635i\,(x-4) \; - 0.0368155i\,(x-4)^2 \; + 0.0076699i\,(x-4)^3 \; + \mathcal{O}(x-4)^4\Big)$$



Triangle systems

- With this approach we can solve all the systems which are in **triangular form**, i.e. those systems for which it has the following form:

$$\frac{d}{dx}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} \star & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & 0 & 0 & \cdots & 0 \\ \star & \star & \star & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \star & \cdots & \star \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{pmatrix}$$

- The idea, hence, is to start from the lowest order in ϵ, solve the first equation, substitute the result in the second and so on. Practically, instead of solving an $n \times n$ system, **we are solving n single equations**.

Coupled systems

- Obtaining a system in a triangular form is not always possible, especially if the problem has an **elliptic nature**. In order to solve the homogeneous system of equation we have to use a generalisation of Frobenius method. Let us consider for example the following system:

$$B_1'(x) = \frac{B_1(x)}{x} - \frac{3B_2(x)}{x} + \left(\frac{1}{2} - \frac{9}{x}\right)$$

$$B_2'(x) = -\frac{2(x-3)B_1(x)}{x(x-9)(x-1)} + \frac{2(5x-9)B_2(x)}{x(x-9)(x-1)} - \frac{648 + (4\pi^2 - 273)x + 27x^2}{12x(x-9)(x-1)}$$

- Let us start from solving the homogeneous part of the system around $x = 1$. To do so let us use the following ansatz:

$$B_1(x) = (x-1)^r \sum_{i=0}^{\infty} a_i\, (x-1)^i$$

$$B_2(x) = (x-1)^r \sum_{i=0}^{\infty} b_i\, (x-1)^i$$

Coupled systems

- Let us substitute and collect order the different orders in $(x-1)$:

$$\mathcal{O}\left(\frac{1}{x-1}\right) : \begin{cases} r\,a_0 = 0 \\ \frac{a_0}{2} - b_0 + r b_0 = 0 \end{cases}$$

$$\mathcal{O}\,(x-1)^0 : \begin{cases} -a_0 + a_1 + r\,a_1 + 3\,b_0 = 0 \\ \frac{1}{16}\left(-11\,a_0 + 8\,a_1 + 34\,b_0 + 16\,r\,b_1\right) = 0 \end{cases}$$

$$\{r = 0,\, a_0 = 2b_0\}$$

$$\mathcal{O}\,(x-1) : \begin{cases} a_0 - a_1 + 2\,a_2 + r\,a_2 - 3\,b_0 + 3\,b_1 = 0 \\ \frac{1}{128}\left(85\,a_0 - 88\,a_1 + 64\,a_2 - 254\,b_0 + 272\,b_1 + 128\,b_2 + 128\,r\,b_2\right) = 0 \end{cases}$$

- Now we can solve all the systems and we find a solution:

$$B_1^{hom}(x) = a_1\left((x-1)^2 - \frac{5(x-1)^3}{4} + \frac{87(x-1)^4}{64} - \frac{91(x-1)^5}{64} + \mathcal{O}(x-1)^6\right)$$

$$B_2^{hom}(x) = a_1\left(-\frac{2(x-1)}{3} + \frac{11(x-1)^2}{12} - \frac{47(x-1)^3}{48} + \frac{97(x-1)^4}{96} - \frac{3161(x-1)^5}{3072} + \mathcal{O}(x-1)^6\right)$$

Coupled systems

- The solution we found to the homogeneous equation depends on **1** arbitrary constant a_1:

$$B_1^{hom}(x) = a_1\left((x-1)^2 - \frac{5(x-1)^3}{4} + \frac{87(x-1)^4}{64} - \frac{91(x-1)^5}{64} + \mathcal{O}(x-1)^6\right)$$

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- However, this is a 2×2 system and so we expected 2 linearly independent solutions! This is because **the ansatz we chose was not general enough**. Let us consider, hence:

$$B_1(x) = (x-1)^r \sum_{i=0}^{\infty} a_i\,(x-1)^i + \log(x-1)\,(x-1)^r \sum_{i=0}^{\infty} c_i\,(x-1)^i$$

$$B_2(x) = (x-1)^r \sum_{i=0}^{\infty} b_i\,(x-1)^i + \log(x-1)\,(x-1)^r \sum_{i=0}^{\infty} d_i\,(x-1)^i$$

Coupled systems

- The procedure is the same, the only difference is that now we collect also powers of $(x-1)^m \log(x-1)$

$$\mathcal{O}\left(\frac{\log(x-1)}{x-1}\right) : \begin{cases} r\, c_0 = 0 \\ \frac{1}{2}\left(c_0 - 2\, d_0 + 2\, r\, d_0\right) = 0 \end{cases} \qquad \mathcal{O}\left(\frac{1}{x-1}\right) : \begin{cases} r\, a_0 + c_0 = 0 \\ \frac{1}{2}\left(a_0 - 2\, b_0 + 2\, r\, b_0 + 2\, d_0\right) = 0 \end{cases}$$

$$\{r = 0,\, c_0 = 2 d_0\}$$

- And the final solution is

$$B_1^{hom}(x) = a_0\left[1 - \frac{x-1}{2} + \frac{9(x-1)^3}{128} + \mathcal{O}(x-1)^4 + \left(\frac{3(x-1)^2}{16} - \frac{15(x-1)^3}{64} + \mathcal{O}(x-1)^4\right)\log(x-1)\right] +$$

$$+a_2\left((x-1)^2 - \frac{5(x-1)^3}{4} + +\mathcal{O}(x-1)^4\right);$$

$$B_2^{hom}(x) = a_0\left[\frac{1}{2} - \frac{x-1}{16} - \frac{7(x-1)^2}{128} + \frac{71(x-1)^3}{1024} + \mathcal{O}(x-1)^4 + \left(-\frac{x-1}{8} + \frac{11(x-1)^2}{64} - \frac{47(x-1)^3}{256} + \mathcal{O}(x-1)^4\right)\log(x-1)\right] +$$

$$a_2\left(-\frac{2(x-1)}{3} + \frac{11(x-1)^2}{12} - \frac{47(x-1)^3}{48} + \mathcal{O}(x-1)^4\right)$$

Coupled systems

- We can organise it in a matrix: $\vec{B}^{hom}(x) = \mathbf{A}(x)\,\vec{c}$. Where $\mathbf{A}_{ij}(x)$ is the i-th solution, where we put all the constant to 0 except the j-th to 1. In this case $\vec{c} = \begin{pmatrix} a_0 \\ a_2 \end{pmatrix}$. We can use again the method of variation of parameters, now all quantities are matrices and vectors.

$$\vec{B}^{part}(x) = \mathbf{A}(x)\,\vec{c}(x) \longrightarrow \frac{\partial}{\partial x}\vec{B}(x) = \mathbf{M}(x)\vec{B}(x) + \vec{g}^{non\ hom}(x)$$

$$\cancel{\mathbf{A}'(x)\,\vec{c}(x)} + \mathbf{A}(x)\,\vec{c}\,'(x) = \cancel{\mathbf{M}(x)\mathbf{A}(x)\,\vec{c}(x)} + \vec{g}^{non\ hom}(x)$$

$$\vec{c}\,'(x) = \mathbf{A}^{-1}(x)\,\vec{g}^{non\ hom}(x) \longrightarrow \vec{B}^{part}(x) = \mathbf{A}(x)\int_0^x \mathbf{A}^{-1}(x')\,\vec{g}^{non\ hom}(x')\,dx'$$

Coupled systems

- We can invert the matrix and perform the integration easily. Once again, by expanding $\vec{g}^{non\ hom}(x)$ around $x = 1$, we have to integrate only terms like $(x-1)^m \ \log^n(x-1)$.
- Finally, the general solution is $\overrightarrow{B}(x) = \mathbf{A}(x)\ \vec{c} + \overrightarrow{B}^{part}(x)$, and the constants are fixed using the boundary conditions:

$$B_1^{hom}(x) = \frac{59}{8} + \frac{\pi^2}{4} + \frac{3}{8}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{139}{384}(x-1)^4 + \mathcal{O}(x-1)^5$$

$$B_2^{hom}(x) = \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) + \frac{1}{4}(x-1)^2 - \frac{25}{96}(x-1)^3 + \frac{155}{576}(x-1)^4 + \mathcal{O}(x-1)^5$$

- Since there is the inversion of a matrix if the system is not in triangular form, it is **computationally more expensive** to solve it. It might be worth to try to decuple the system so that the resolution is quicker.
- In principle with this approach **we could solve any system of differential equations**. Facing all types of physical problems, including the elliptic ones.

Boundary Conditions

- The first one is to **provide the value of the master integral** in a regular or a pseudo-threshold point;

$$f(x) = c\left(1 - \frac{x}{5} - \frac{3x^2}{50} - \frac{11x^3}{750} + \mathcal{O}(x^4)\right) + \left(\frac{1}{2}x - \frac{7x^2}{40} + \frac{2x^3}{75} + \mathcal{O}(x^4)\right) \qquad f(0) = 1$$

- The second is to **impose the regularity of the solution** in a pseudo-threshold point;

$$f(x) = c\, x^{-1/2}\left(1 + \frac{x}{8} + \frac{3x^2}{128} + \frac{5x^3}{1024} + \mathcal{O}(x^4)\right) + \left(\frac{1}{2} + \frac{x}{12} + \frac{x^2}{60} + \frac{x^3}{280} + \mathcal{O}(x^4)\right)$$

- A third possibility is to **impose the coefficient of the divergent part**, such as $\log x$ or $1/x^m$.

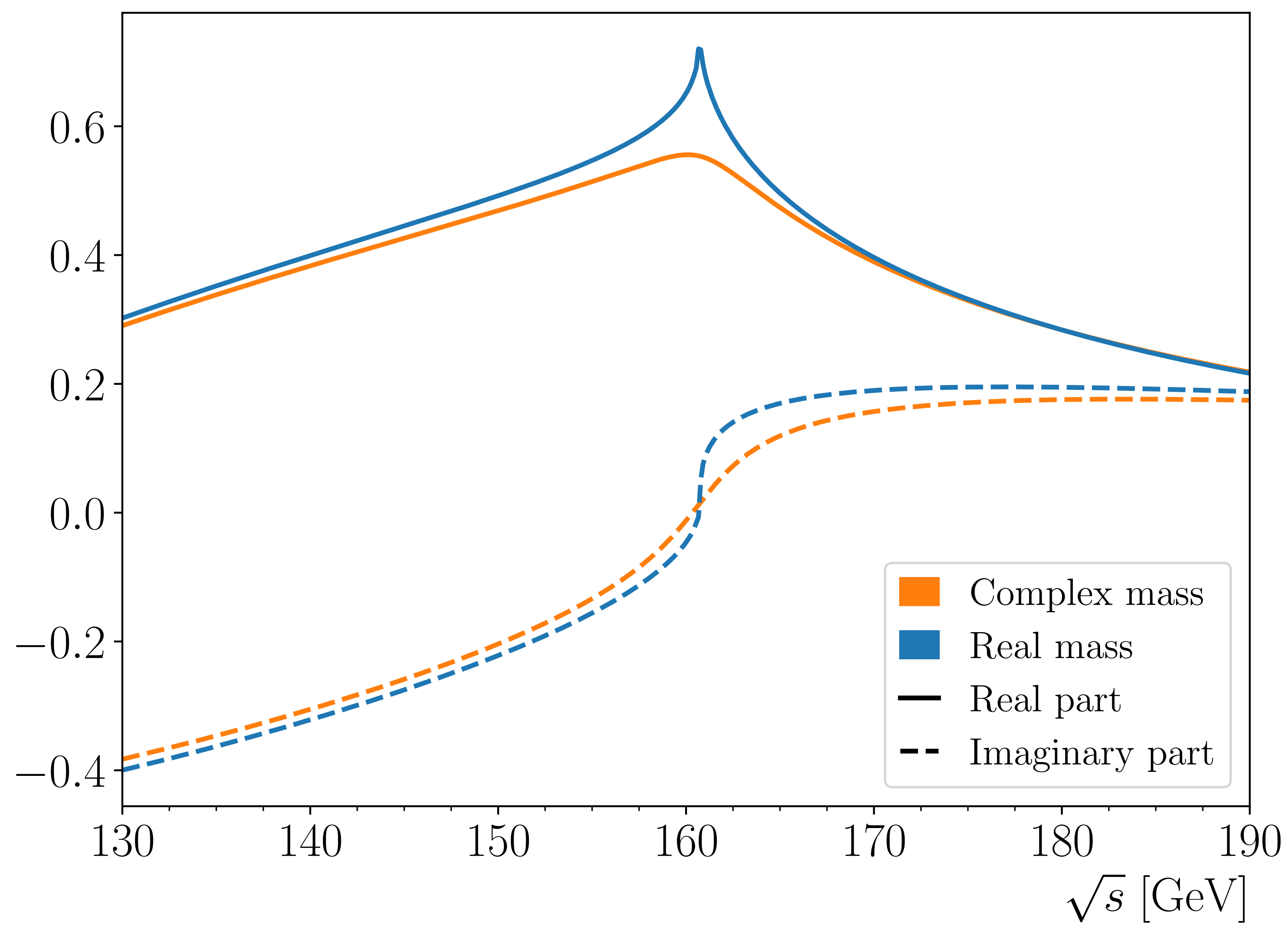
$$f(x) = \frac{i\pi - ic + \log 2}{\sqrt{x-4}} + \mathcal{O}(x-4) \qquad f(0) = \frac{\log 2}{\sqrt{x-4}} + \mathcal{O}(x-4)$$

- The boundary conditions are, in general, **not trivial to obtain**. Some common techniques are the Auxiliary mass flow method, direct integration outside of the physical region, Monte-Carlo integration, expansion by region.

Complex Mass Scheme

- When working on EW calculations, we have to deal with intermediate unstable particles, such as W and Z. In this case, it is useful to perform the calculations in the **complex-mass scheme**;
- For these particles we consider their mass to be complex-valued:

$$\mu_V^2 = m_V^2 - i\Gamma_V m_V$$



- The complex mass scheme **regularises** the divergences coming from the tree-level propagators, while preserving gauge invariance.

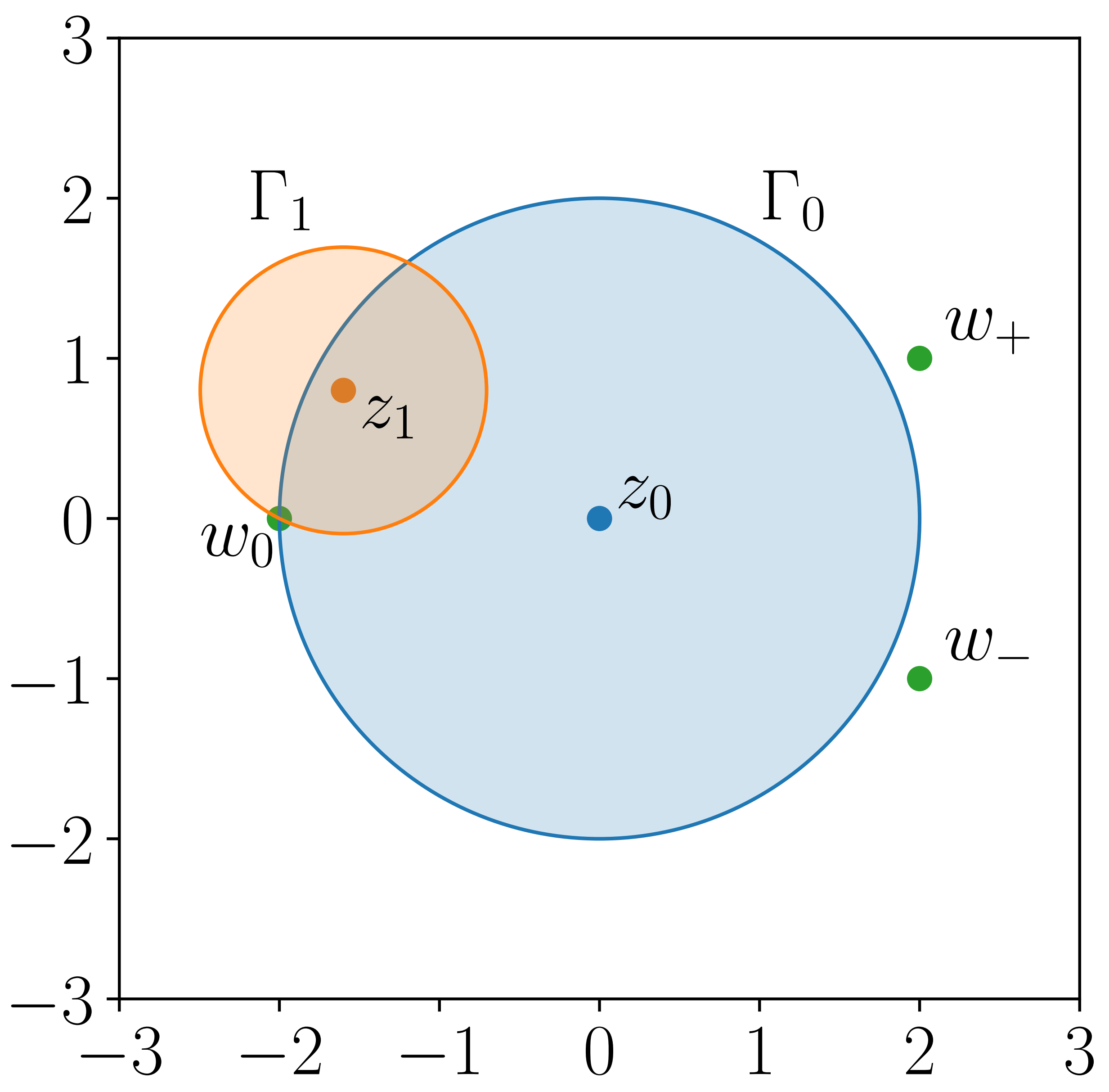
$$\frac{1}{s - m_V^2 + i\delta}$$

- However, it requires all the masses to be complex-valued, included the ones in the Feynman integrals. If we utilise **adimensional variables**, they become complex-valued as well:

$$x = \frac{s}{m_V^2} \rightarrow \frac{s}{\mu_V^2}$$

Analytic continuation

- As we saw, the analytic continuation must be discussed in the entire complex plane
- Power series have a limited **radius of convergence**.
- The radius is determined by the position of the **nearest singularity**.



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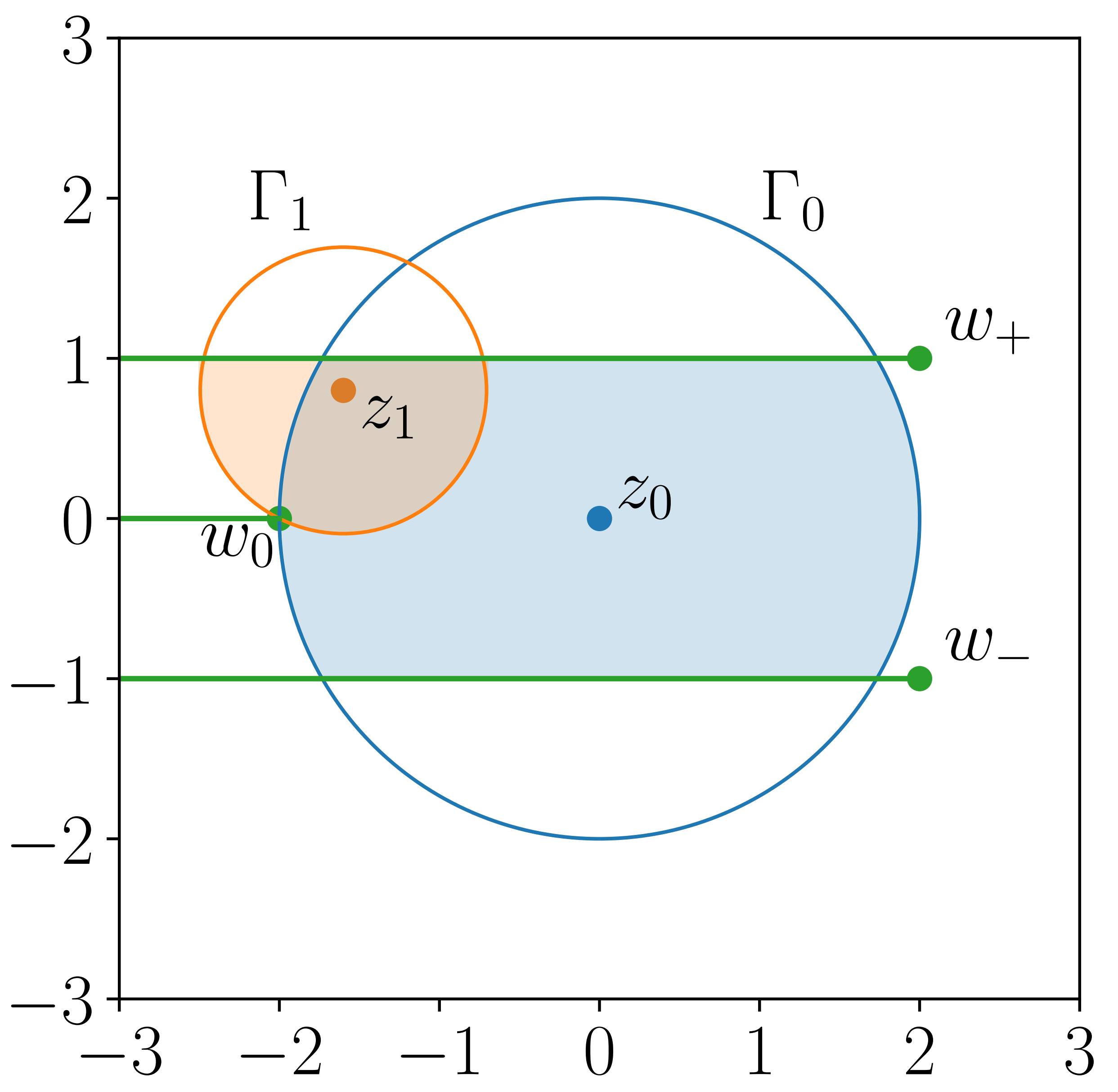

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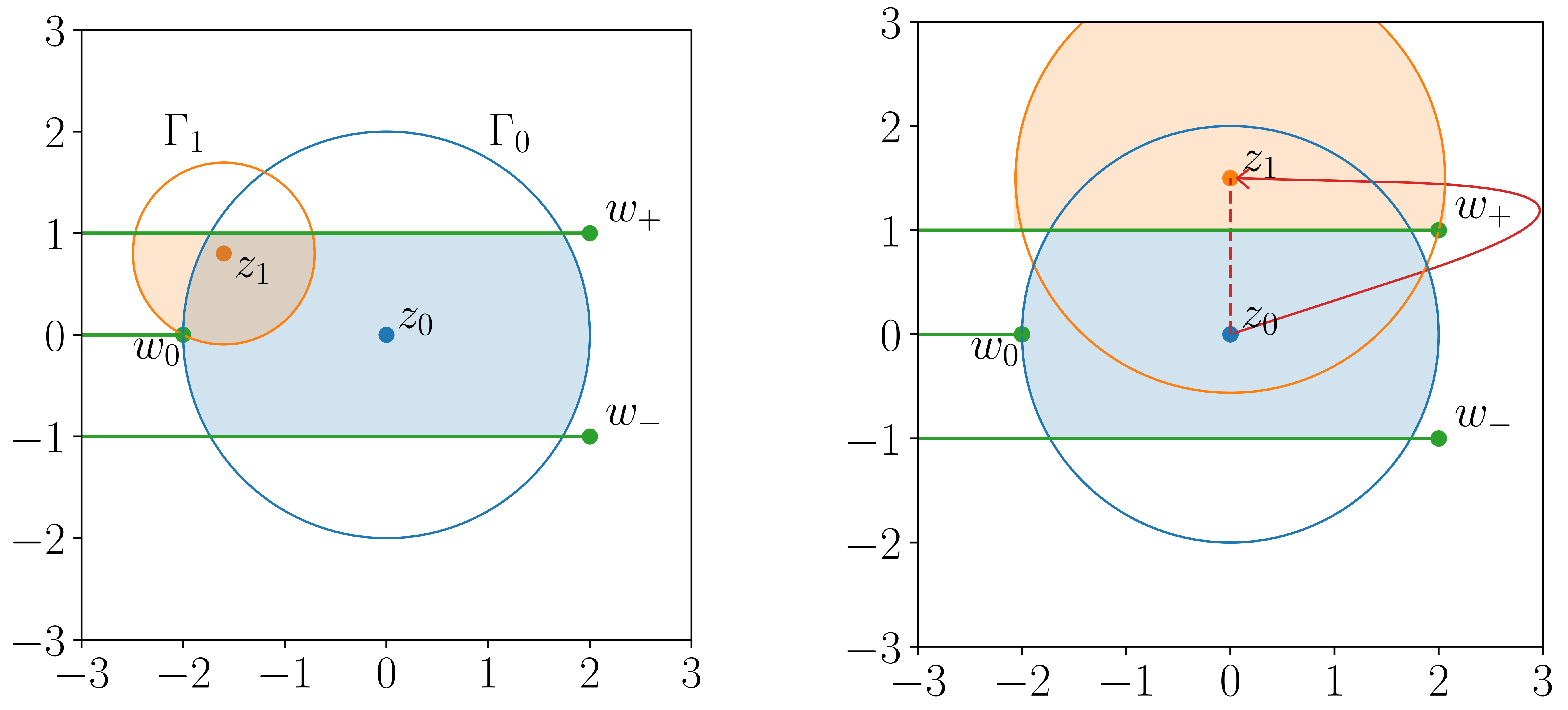
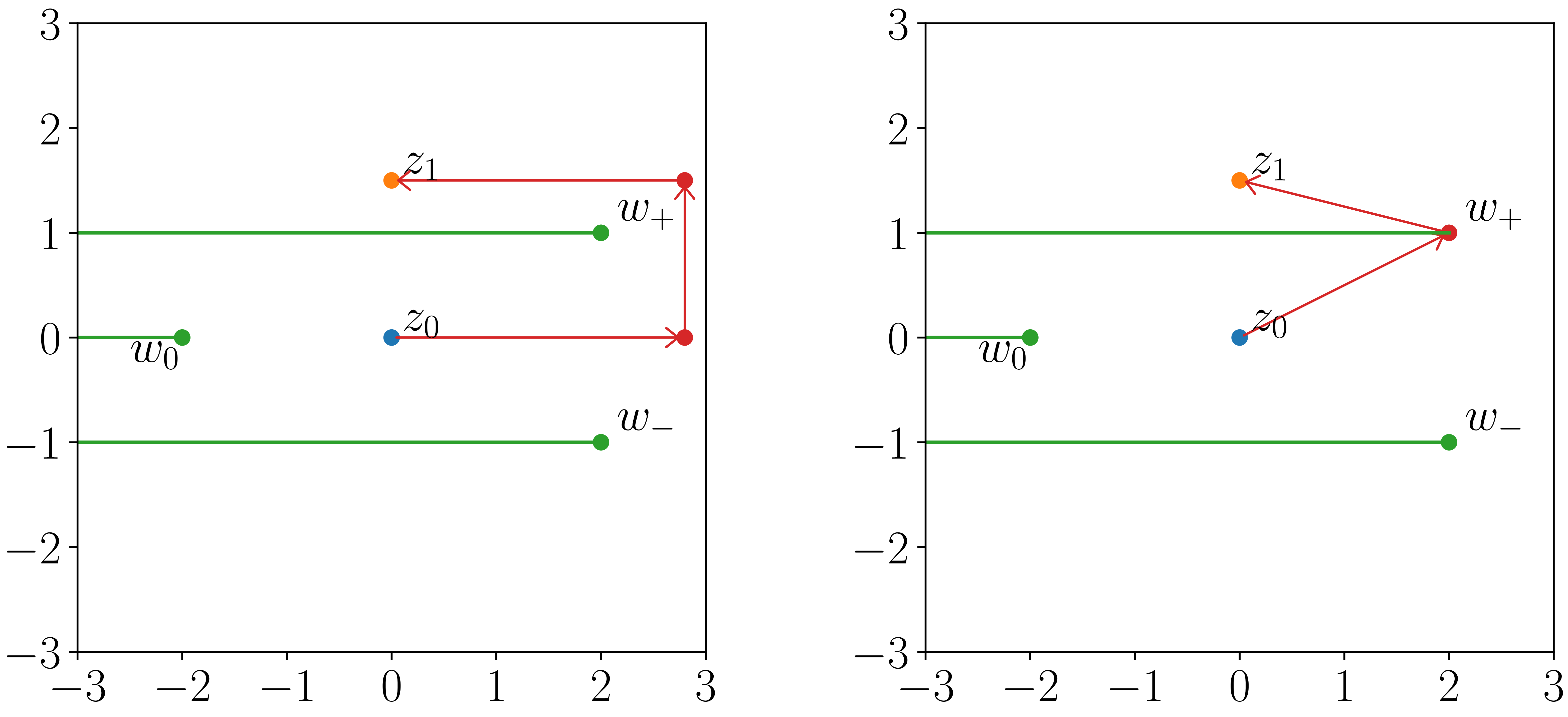


Taylor vs Logarithmic

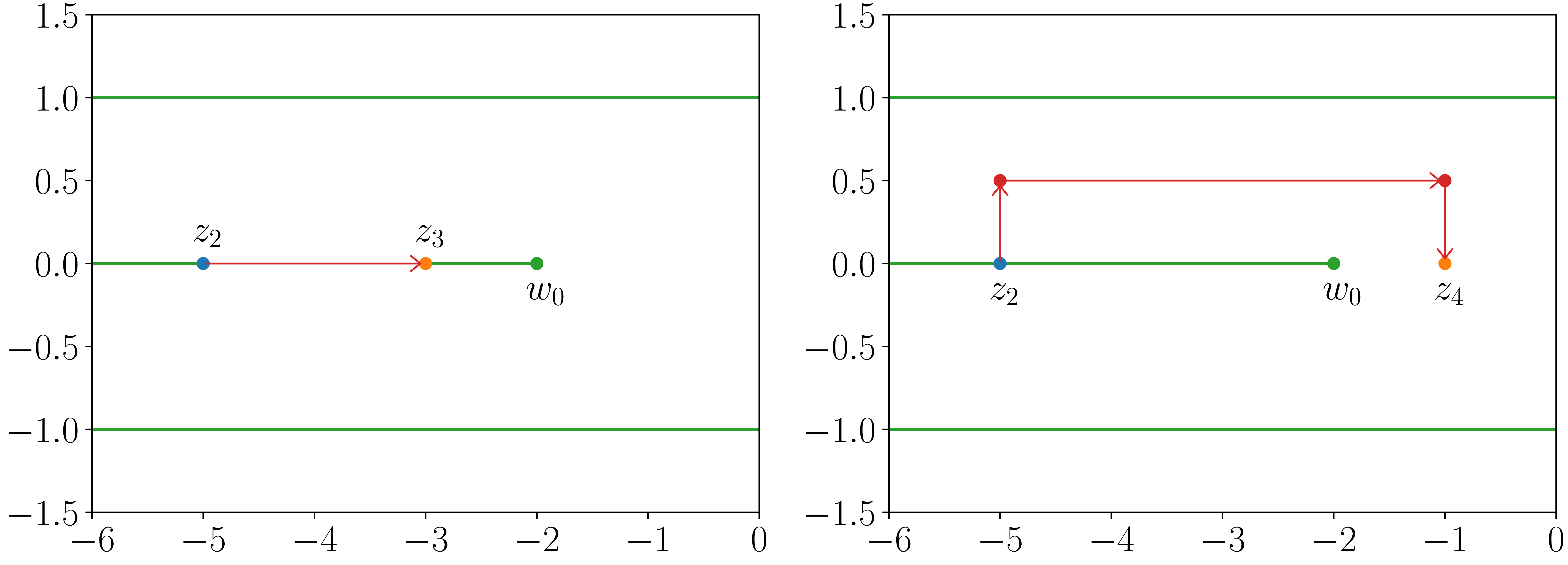
- When moving along an horizontal line, the Feynman prescription plays an important role



Analytic continuation

- When moving along an horizontal line, the **Feynman prescription** plays an important role

$$\frac{1}{s - m_V^2 + i\delta}$$



Automatic packages

- This idea was first introduced in the study of Higgs+jet production at 2-loop *[F.Moriello, arXiv:1907.13234]*.
- The first publicly available Mathematica package implementing this technique is **DiffExp** *[M.Hidding, arXiv:2006.05510]*. The main limitation of DiffExp is the fact that it can work only with **real-valued variables**. For this reason, it is suitable with QCD calculations, but not for EW ones.

- Another Mathematica implementation is in the package **SeaSyde** *[**TA**, R. Bonciani, S. Devoto, N. Rana, A. Vicini, arXiv:2205.03345]*. For the first time we introduced the algorithm for the analytic continuation in the complex plane, thus allowing it to be used in **EW calculations**. For example it has been used for the calculation of NNLO mixed QCD-EW corrections to the Drell-Yan process.

SEASYDE

Automatic packages

- A third independent implementation is in the Mathematica package **AMFlow** *[X. Liu, Y. Ma, arXiv:2201.11669]*. In particular, they use the **auxiliary mass flow method** for automatically obtaining the boundary conditions.

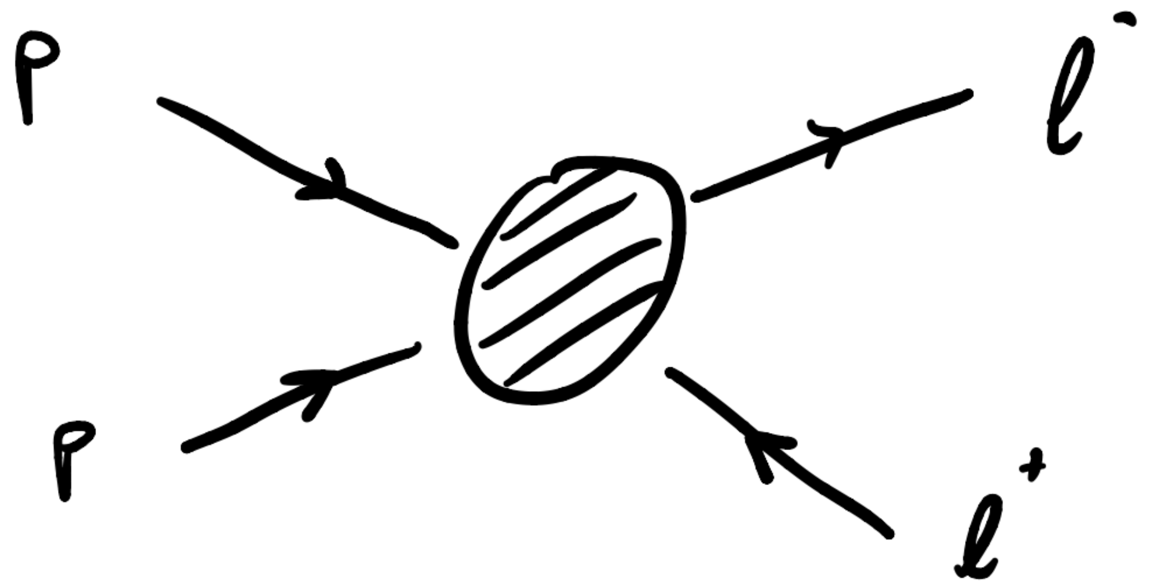
$$I_{aux}(\alpha_i; s_j, d, \eta) = \int \prod_{k=1}^{l} \frac{d^d q_k}{i\pi^{d/2}} \frac{1}{[\mathcal{D}_1 - \eta]^{\alpha_1} \ \ldots \ [\mathcal{D}_n - \eta]^{\alpha_n}}$$

- In the limit $\eta \to \infty$ the integrals simplify and thus they can be easily evaluated analytically. Then we can write down the differential equation w.r.t. η, and, finally, recover the desired integral by evolving I_{aux} from ∞ to $i0^-$

$$\frac{\partial}{\partial \eta} \vec{I}_{aux} = A(\eta) \vec{I}_{aux} \qquad\qquad I = \lim_{\eta \to i0^-} I_{aux}(\eta)$$

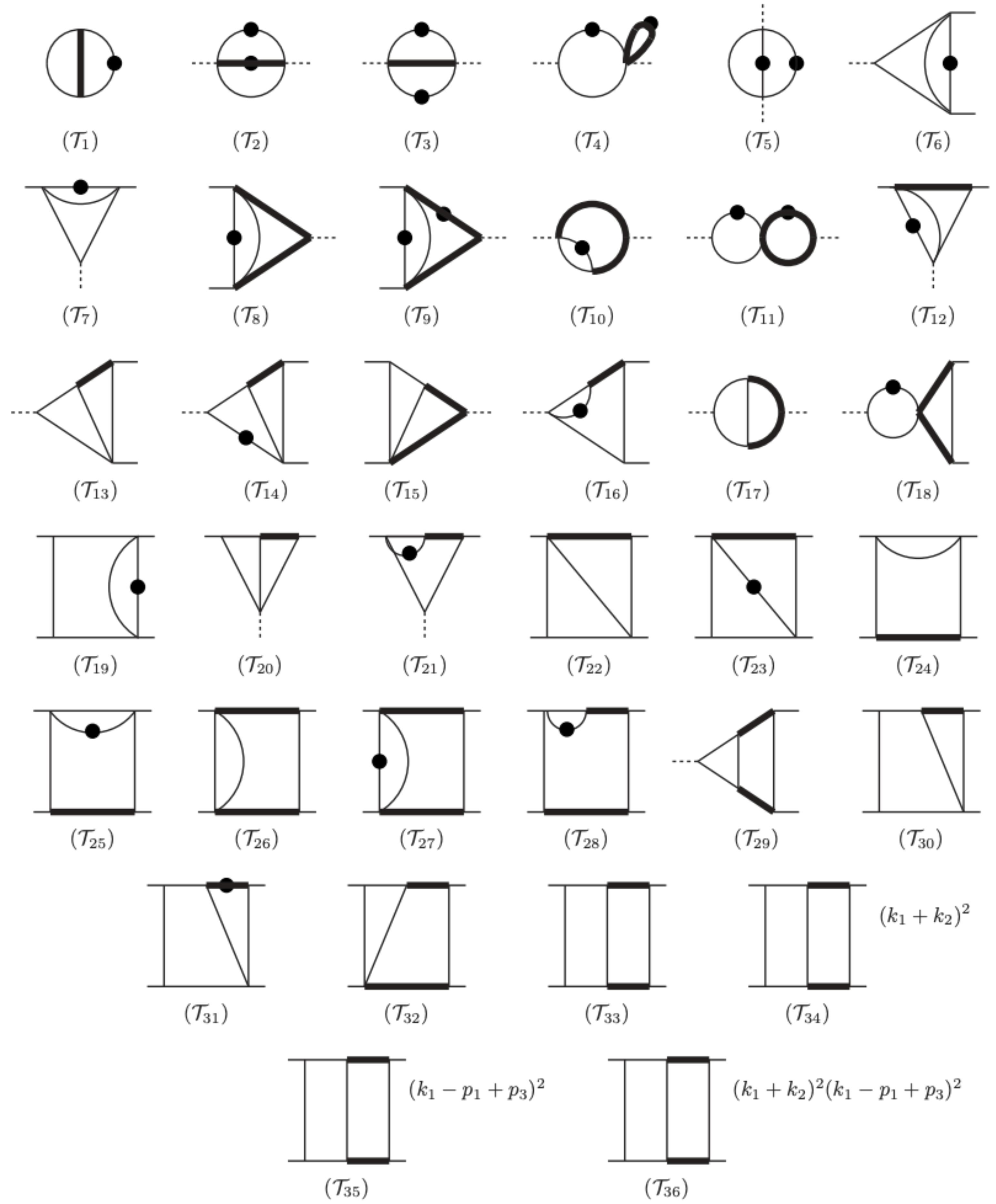
- All three packages can solve **all type of problems**, including elliptic ones.
- A fourth group is working on a C++ implementation **LoopTransport** *[T. Neumann]*, however, this is not public yet.

Neutral-current Drell-Yan



- The first physical application of SeaSyde was for the calculation of the mixed QCD-EW corrections for the **Neutral-Current Drell-Yan** *[**TA**, R. Bonciani, S. Devoto, N.Rana, A.Vicini, arXiv:2201.01754]*

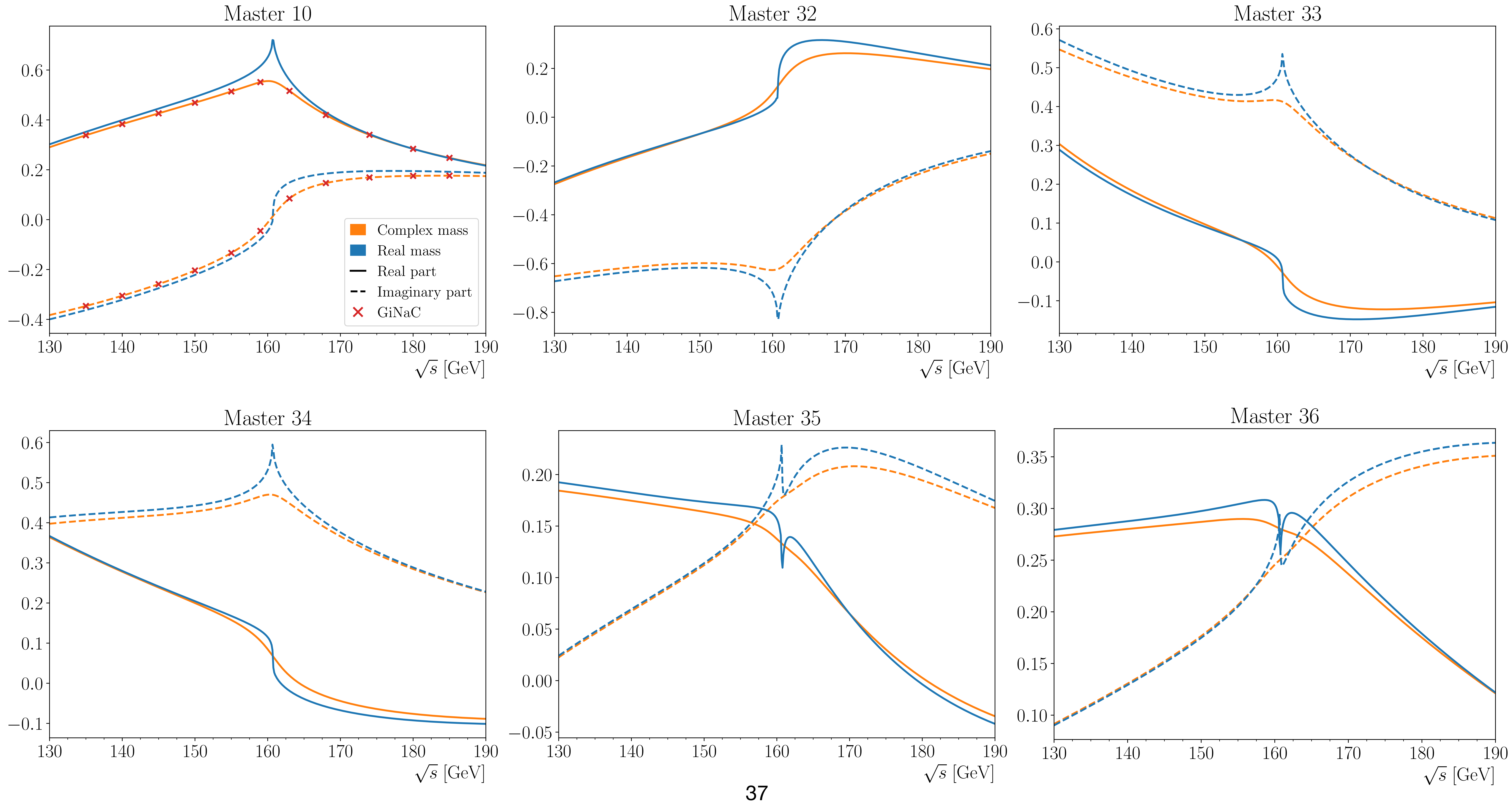
$$\sigma^{(0,0)}$$
$$+\alpha_S\,\sigma^{(1,0)}+\alpha\,\sigma^{(0,1)}$$
$$+\alpha_S^2\,\sigma^{(2,0)}+\alpha_S\,\alpha\,\sigma^{(1,1)}+\alpha^2\,\sigma^{(0,2)}+\dots$$



- We had **36 Master Integrals** with 2 internal (complex-)massive lines. For 31 out of 36 we had an expression in terms of Generalised PolyLogarithms (GPLs). However, for 5 of them we only had an expression in terms of Chen-Goncharov integrals, which are not suitable for a numerical evaluation.

Neutral-current Drell-Yan

- 31 masters provide cross checks with analytic results, 5 are a prediction;
- complex mass scheme smoothens the behaviour at threshold.

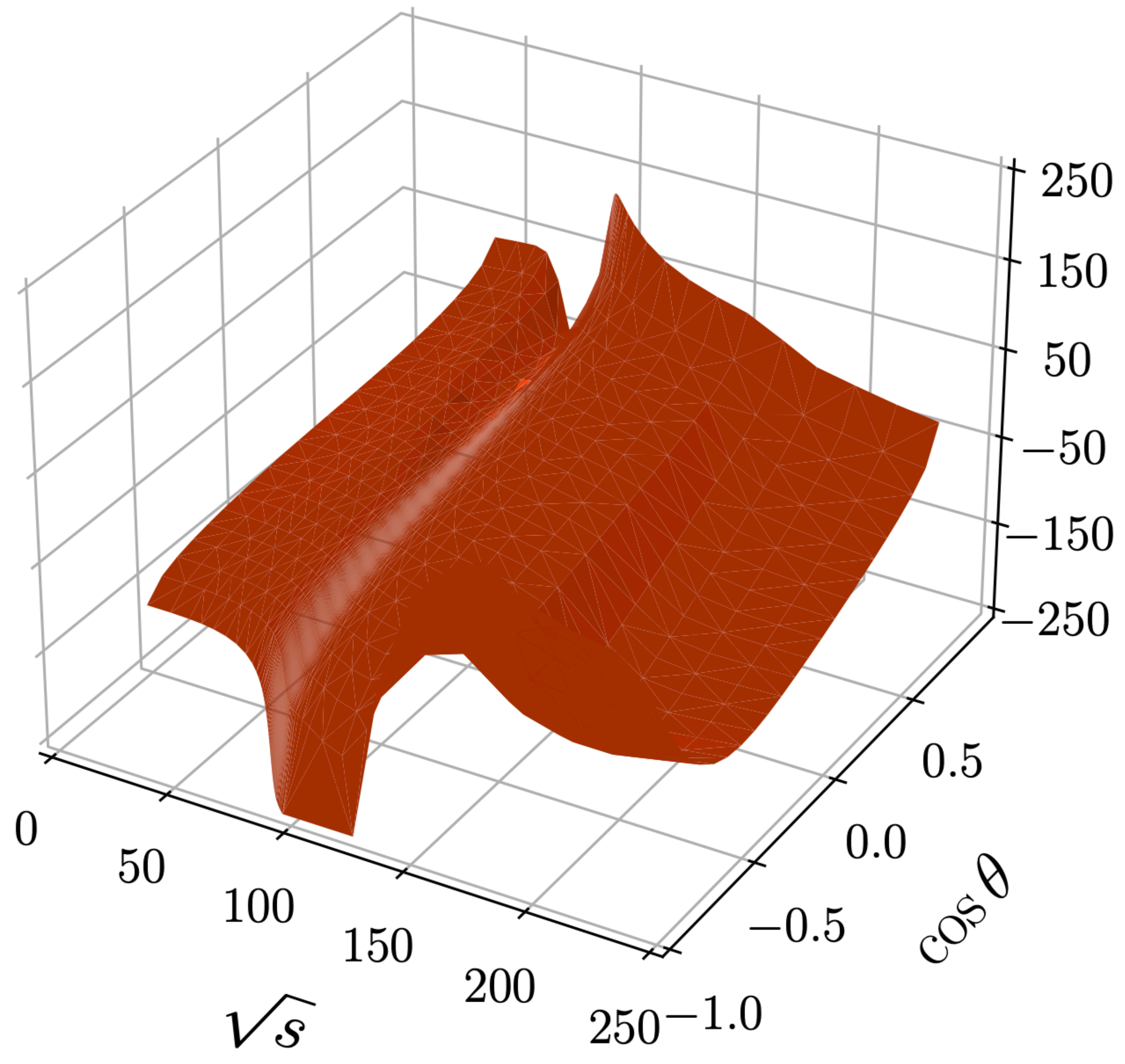


Neutral-current Drell-Yan

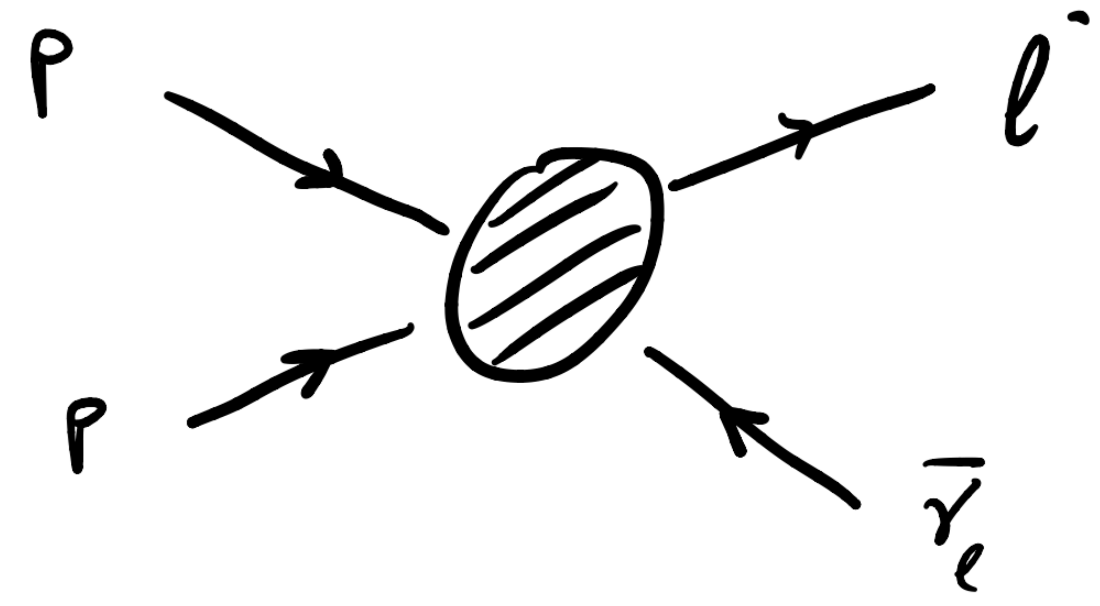
- The boundary conditions are imposed in the **euclidean region**, outside of the physical one. Transporting the solution to the physical region requires different times depending on the number of terms in the series required.

Number of terms	Precision	Time
50 terms	10^{-14}	~14 min
75 terms	10^{-19}	~26 min
100 terms	10^{-25}	~50 min
125 terms	10^{-33}	~75 min
150 terms	10^{-40}	~90 min

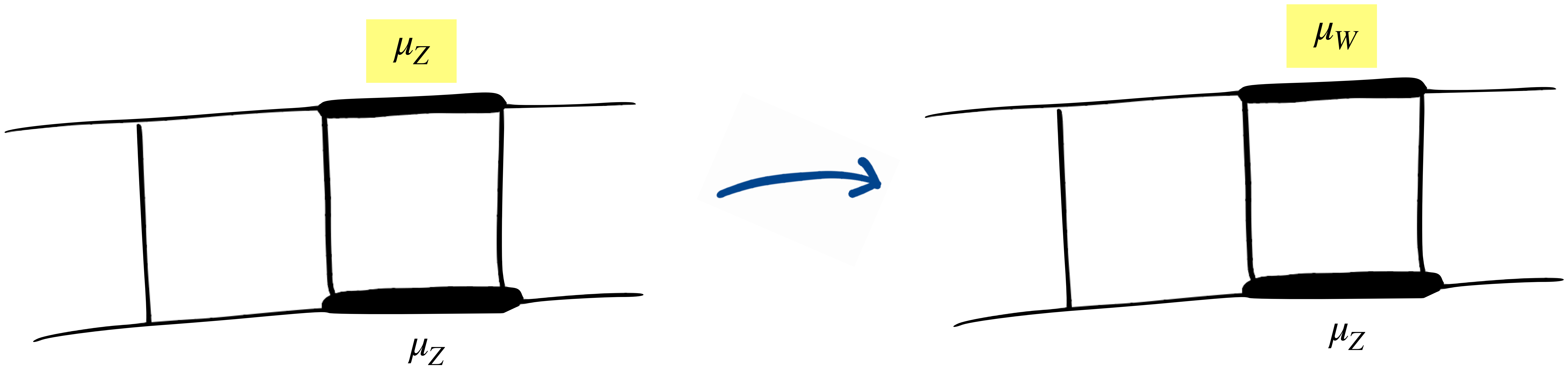
- Given the execution time it is impossible to implement directly in a Monte-Carlo generator. For this reason, we created a grid for the final correction and then, thanks to its smoothness we can interpolate it. The grid consist of 3250 point spanning $\sqrt{s} \in [40 \text{ GeV}, 13 \text{ TeV}]$ and $\cos\theta \in [-1,1]$.



Charged-current Drell-Yan



- This process, even if similar to the previous one, is more complicated because now we have integrals with **two different internal massive lines**.
- Those integrals belong to a two-loop box integral family. In this family we have **56 masters**. We could proceed in the same way as before, that is find the boundary conditions and then use the differential equations w.r.t. s and t to create a grid which covers the entire phase space.
- Another possibility is to write down the differential equations w.r.t. one of the two masses and use the grid of the neutral current Drell-Yan as a boundary condition.



Conclusion

- The method of differential equations, and in particular its semi-analytical approach, is a powerful technique for evaluating Feynman integrals. In particular, it is a method easy to automatise and completely general. Moreover, we can easily control the numerical precision of the result;
- Its main problem right now is the speed, mainly due to the fact that all its implementations are in Mathematica. For this reason we have to rely on pre-computed grids which are then interpolated in Monte-Carlo generator;
- We implemented the method in the publicly available Mathematica package SeaSyde, which can handle arbitrary internal complex masses;
- The method has been already applied in the calculation of the mixed QCD-EW corrections to the Neutral current Drell-Yan and to the charged current case (not yet public).

THANK YOU