Deep Inelastic Scattering

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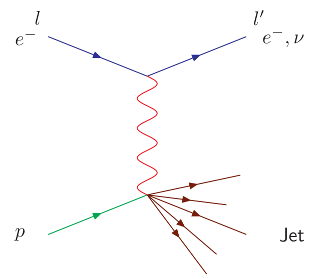
Outline

Kinematics of Deep Inelastic Scattering

What is inside nucleons?

- Basic Idea: Smash a well known probe on a nucleon or nucleus in order to try to figure out what it is made of.



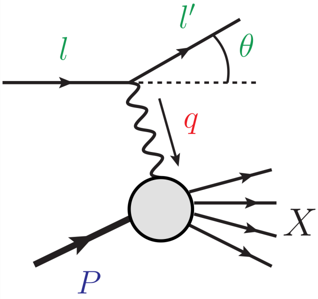
- Electrons are well suited for that purpose because their interactions are well understood.
- Deep Inelastic Scattering: Collision between an electron and a nucleon or nucleus by exchange of a virtual vector boson (photon, Z, W).
- Variant: Collisions with a neutrino (then only Z, W are possible).

Kinematic Variables

- We consider inclusive DIS where we sum over all hadronic final states X:

$$e^-(l) + N(p) \to e^-(l') + X(p_x)$$

- On-shell conditions: $p^2 = M^2,\ l^2 = l'^2 = m^2$
- Measure energy and polar angle of scattered electron (E', θ)
- Other invariants of the reaction:
  - $Q^2 = -q^2 = -(l - l')^2 > 0$, the square of the momentum transfer
  - $\nu = p \cdot q / M^2 \overset{\text{lab}}{=} E_l - E_{l'}$
  - $x = Q^2/(2p \cdot q)$, the Bjorken scaling variable
  - $y = p \cdot q / p \cdot l \overset{\text{lab}}{=} (E_l - E_{l'})/E_l$ the inelasticity parameter
  - $s = (l + p)^2$, the cms energy
- The 'lab' frame designates the rest frame of the nucleon $p = (M, 0, 0, 0)$.



Kinematic Variables

- There are only two independent variables to describe the kinematics of inclusive DIS (up to trivial azimuthal angle dependence):

$$(E', \theta) \quad \text{or} \quad (x, Q^2) \quad \text{or} \quad (x, y) \quad \text{or...}$$

- Relation between Q^2, x, and y:

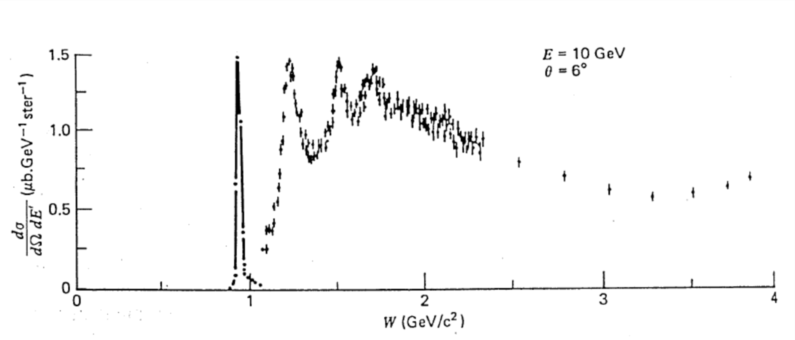
$$\begin{aligned} Q^2 &= (2p \cdot l) \left(\frac{Q^2}{2p \cdot q} \right) \left(\frac{p \cdot q}{2p \cdot l} \right) \\ &= (2p \cdot l)\, x\, y = (s - M^2 - m^2)\, x\, y \end{aligned}$$

- Invariant mass W of the hadronic final state X: (also called missing mass since only outgoing electron is measured)

$$\begin{aligned} W^2 = M_X^2 = (p+q)^2 &= M^2 + 2p \cdot q + q^2 \\ &= M^2 + \frac{Q^2}{x} - Q^2 = M^2 + Q^2 \frac{1-x}{x} \end{aligned}$$

- elastic scattering: $W = M$, $x = 1$
- inelastic: $W \geq M + m_\pi$, $x < 1$

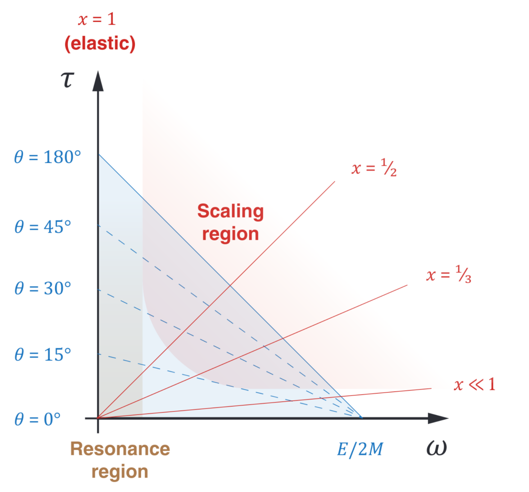
The DIS Cross Section as Function of W



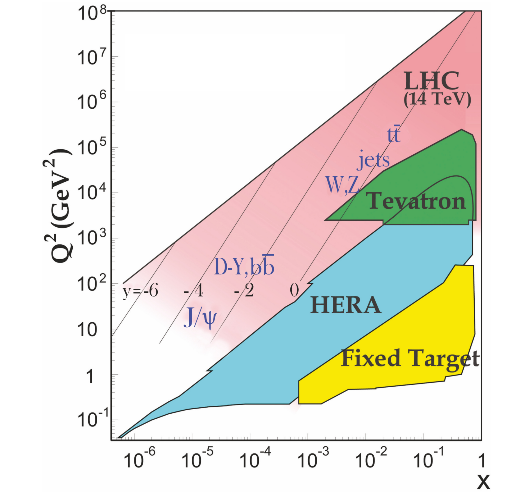
Data from SLAC, where the elastic peak at $W = M$ has been reduced by a factor 8.5. [taken from Halzen, Martin, Quarks and Leptons]

- Elastic peak: $W = M$, $x = 1$ (proton does not break up)
- Resonances: $W = M_\Delta$, $1/x = ...$ (resonances of the proton are exited, note that there is also non-resonant background)
- Inelastic region: $W \gtrsim 1.8 GeV$ (complicated multiparticle final state results in a smooth distribution) (note that there are also charmonium and bottonium resonances at higher energies)

Phase Space of DIS

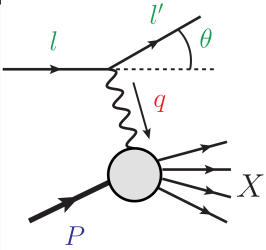


Phase Space of DIS



Cross sections for inclusive DIS

The cross section for inclusive $ep \to eX$



- We consider inclusive DIS where we sum over all hadronic final states X:

$$e^-(l) + N(p) \to e^-(l') + X(p_x)$$

  - The amplitude A is proportional to the interaction of a leptonic current j with a hadronic current J:

$$A \sim \frac{1}{q^2}\, j^\mu\, J_\mu$$

  - The leptonic current can be readily evaluated in perturbative QED:

$$j^\mu = \langle l', s_{l'} \left| \hat{j}^\mu \right| l, s_l \rangle = \bar{u}(l', s_{l'})\gamma^\mu u(l, s_l)$$

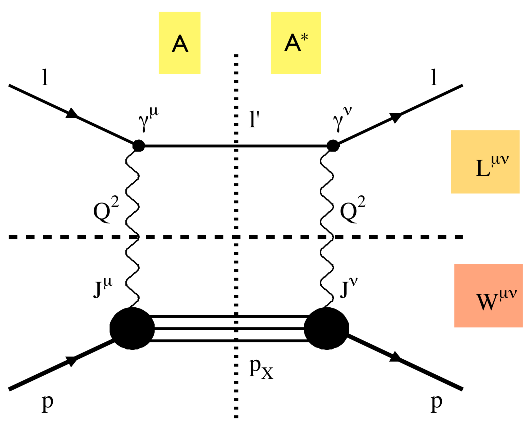
  - The hadronic current is non-perturbative and depends on the multi-particle final state over which we sum:

$$J^\mu = \langle X, s_X \left| \hat{J}^\mu \right| P, s_P \rangle$$

The cross section for inclusive $ep \to eX$

The cross section which is proportional to the amplitude squared can be factored into a leptonic and a hadronic piece:

$$d\sigma \sim |A|^2 \sim L_{\mu\nu} W^{\mu\nu}$$



- $L_{\mu\nu}$: Leptonic tensor (calculable in perturbation theory)
- $W^{\mu\nu}$: Hadronic tensor (non-perturbative)

The cross section for inclusive $ep \to eX$

$$\mathrm{d}\sigma = \sum_X \frac{1}{F} \langle |A_X|^2 \rangle_{\text{spin}} \mathrm{d}Q_X \frac{\mathrm{d}^3 l'}{(2\pi)^3 2E'} = \frac{1}{F} \left[\frac{e^4}{(q^2)^2} L_{\mu\nu} W^{\mu\nu} 4\pi \right] \frac{\mathrm{d}^3 l'}{(2\pi)^3 2E'}$$

- With the Moller flux: $F = 4\sqrt{(l \cdot p)^2 - m_l^2 M^2}$
- The phase space of the hadronic final state X with N_X particles:

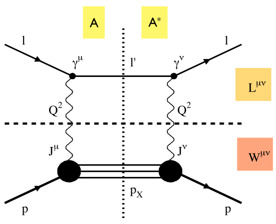
$$\mathrm{d}Q_X = (2\pi)^4 \delta^{(4)}(p + q - p_X) \prod_{k=1}^{N_X} \frac{\mathrm{d}^3 p_k}{(2\pi)^3 2E_k} = (2\pi)^4 \delta^{(4)}(p + q - p_X) \mathrm{d}\Phi_X$$

- The amplitude with final state X:

$$A_X = \frac{e^2}{q^2} \left[\bar{u}(l') \gamma^\mu u(l) \right] \langle X | J_\mu(0) | N(p) \rangle, \quad A_X^* = \frac{e^2}{q^2} \left[\bar{u}(l) \gamma^\nu u(l') \right] \langle N(p) | J_\nu(0) | X \rangle$$

The cross section for inclusive $ep \to eX$

$$\mathrm{d}\sigma = \sum_X \frac{1}{F}\langle |A_X|^2\rangle_{\text{spin}} \mathrm{d}Q_X \frac{\mathrm{d}^3 l'}{(2\pi)^3 2E'} = \frac{1}{F}\left[\frac{e^4}{(q^2)^2} L_{\mu\nu} W^{\mu\nu} 4\pi\right] \frac{\mathrm{d}^3 l'}{(2\pi)^3 2E'}$$



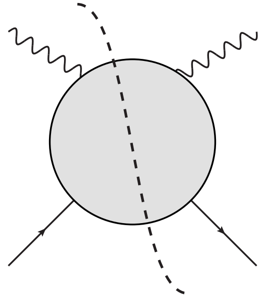
The leptonic tensor is given by:

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2}\sum_{s_l}\sum_{s_{l'}} \bar{u}(l')\gamma_\mu u(l)\bar{u}(l)\gamma_\nu u(l') \\ &= \frac{1}{2}\mathrm{tr}\left[\gamma_\mu(\not{l} + m_l)\gamma_\nu(\not{l}' + m_l)\right] \\ &= 2\left[l_\mu l'_\nu + l_\nu l'_\mu - g_{\mu\nu}(l \cdot l' - m_l^2)\right] = L_{\nu\mu} \end{aligned}$$

The hadronic tensor is defined as:

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi}\sum_X \int \mathrm{d}\Phi_X (2\pi)^4 \delta^{(4)}(p + q - p_X)\langle N(p)\left|J_\nu^\dagger(0)\right| X\rangle\langle X \left|J_\mu(0)\right| N(p)\rangle \\ &= \int \mathrm{d}y\, e^{iqy}\, \langle N(p)\left|\left[J_\nu^\dagger(y), J_\mu(0)\right]\right| N(p)\rangle \end{aligned}$$

The hadronic tensor and structure functions



- The hadronic tensor $W_{\mu,nu}(p,q)$ cannot be calculated in perturbation theory.
- BUT: we can write down the most general covariant expression for $W_{\mu,nu}(p,q)$.
- Also other symmetries like current conservation, parity, etc. have to be respected (depending on the interaction).

  - All possible tensors using p^μ, q^ν:

$$g_{\mu\nu}\,, p_\mu p_\nu\,, q_\mu q_\nu\,, p_\mu q_\nu + p_\nu q_\mu, \epsilon_{\mu\nu\rho\sigma} p^\rho q^\sigma\,, p_\mu q_\nu - p_\nu q_\mu$$

  - For a (spin-averaged) nucleon, the most general covariant expression for $W_{\mu,\nu}(p,q)$ is:

$$W_{\mu,\nu}(p,q) = -g_{\mu\nu} W_1 + \frac{p_\mu p_\nu}{M^2} W_2 - i\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{M^2} W_3$$
$$+\frac{q_\mu q_\nu}{M^2} W_4 + \frac{p_\mu q_\nu + p_\nu q_\mu}{M^2} W_5 + \frac{p_\mu q_\nu - p_\nu q_\mu}{M^2} W_6$$

  - The structure functions W_i can depend on the Lorentz-invariants M^2, Q^2, x and internal masses.

The hadronic tensor and structure functions

$$W_{\mu,\nu}(p,q) = -g_{\mu\nu}W_1 + \frac{p_\mu p_\nu}{M^2}W_2 - i\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{M^2}W_3 + \frac{q_\mu q_\nu}{M^2}W_4 + \frac{p_\mu q_\nu + p_\nu q_\mu}{M^2}W_5 + \frac{p_\mu q_\nu - p_\nu q_\mu}{M^2}W_6$$

- $W_3 = 0$ and $W_6 = 0$ for parity conserving currents.
- W_6 does not contribute to the cross section.
- Since $q^\mu L_{\mu\nu} \sim m_l^2$ the structure functions W_4 and W_5 contribute proportional to the lepton mass squared in the cross section (usually negligible).
- Parity and time reversal symmetry of QCD imply: $W_{\mu\nu} = W_{\nu\mu}$

The hadronic tensor and structure functions

$$W_{\mu,\nu}(p,q) = -g_{\mu\nu} W_1 + \frac{p_\mu p_\nu}{M^2} W_2 - i\epsilon_{\mu\nu\rho\sigma} \frac{p^\rho q^\sigma}{M^2} W_3 + \frac{q_\mu q_\nu}{M^2} W_4 + \frac{p_\mu q_\nu + p_\nu q_\mu}{M^2} W_5 + \frac{p_\mu q_\nu - p_\nu q_\mu}{M^2} W_6$$

- Current conservation at the hadronic vertex implies $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$:

$$W_5 = -\frac{p \cdot q}{q^2} W_2, \qquad W_4 = \left(\frac{p \cdot q}{q^2}\right)^2 W_2 + \frac{M^2}{q^2} W_1$$

- For the spin-averaged hadronic tensor with photon exchange we are left with 2 independent structure functions:

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \frac{1}{M^2}\left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right) W_2$$

The cross section for inclusive $e\,p \to e\,X$

The DIS cross section in the nucleon rest frame reads (photon exchange, neglecting m_l):

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\,\mathrm{d}\Omega'} = \frac{\alpha^2}{4ME^2\sin^4(\theta/2)}\left[2W_1(x,Q^2)\sin^2(\theta/2) + W_2(x,Q^2)\cos^2(\theta/2)\right]$$

Usually used:

$$\{F_1, F_2, F_3\} = \left\{W_1, \frac{Q^2}{2xM^2}, \frac{Q^2}{xM^2}W_3\right\}$$

The DIS cross section in terms of Lorentz-invariants (photon exchange, neglecting m_l):

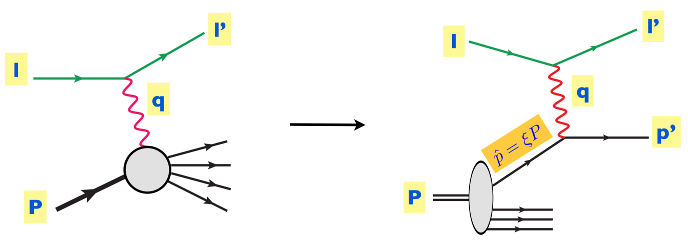
$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\,x\mathrm{d}\,y} = \frac{4\pi\alpha^2 S}{Q^4}\left[xy^2F_1(x,Q^2) + (1-y-x\,y\,M^2/S)F_2(x,Q^2)\right]$$

The naive parton model

The naive parton model

- The naive parton model assumes that the nucleon is a collection of point-like constituents called partons.
- At high momentum (infinite momentum frame) the partons are free (non-interacting). Therefore the interaction of one parton wih the electron does not affect the other partons.
- In the infinite momentum frame, we have $P \sim (E_P, 0, 0, E_P)$ with $E_P \gg M$. The partons are moving parallel with the proton carrying a fraction ξ of its momentum $\hat{p} = \xi P$.

The naive parton model

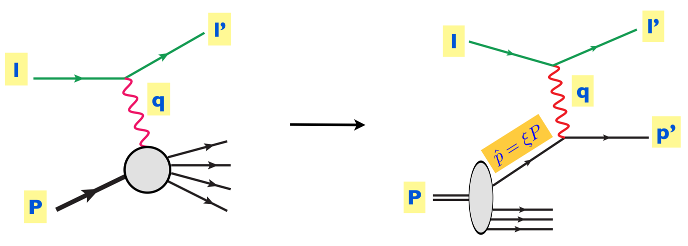


$$\mathrm{d}\sigma(e(l) + N(P) \to e(l') + X(p_X)) = \sum_i \int_0^1 f_i(\chi)\mathrm{d}\sigma(e(l) + i(\chi P) \to e(l') + i(p'))$$

We have replaced the scattering off the complicated nucleon with:

- The incoherent sum over all possible partonic processes.
- Parton densities: $f_i(\xi)\mathrm{d}\xi$ describes the number of parton i with momentum fraction in $[\xi, \xi + \mathrm{d}\xi]$.
- Elastic scattering off point-like partons.

The naive parton model



$$W^{\mu\nu}(P,q) = \sum_i \int_x^1 \frac{\mathrm{d}\xi}{\xi} f_i(\xi)\hat{w}_i^{\mu\nu}(\xi P, q)$$

- $\hat{w}_i^{\mu\nu}$: partonic tensor calculable perturbatively
- $f_i(\xi)$: parton distributions, non-perturbative but universal

Structure functions in the parton model

We will calculate the contribution of a spin-1/2 parton of type i to the partonic tensor:

$$\begin{aligned}\hat{w}_i^{\mu\nu} &= \frac{1}{2}\int \frac{\mathrm{d}^4 p'}{(2\pi)^4}\delta(p'^2)(2\pi)^4\delta^{(4)}(\xi P + q - p')\langle \xi P \left| J^{\mu,\dagger}(0) \right| p'\rangle\langle p' \left| J^{\nu}(0) \right| \xi P\rangle \\ &= \frac{1}{2}\delta((\xi P + q)^2)\langle \xi P \left| J^{\mu,\dagger}(0) \right| \xi P + q\rangle\langle \xi P + q \left| J^{\nu}(0) \right| \xi P\rangle\end{aligned}$$

This is the same calculation as we have already done for the leptonic tensor:

$$\begin{aligned}\hat{w}_i^{\mu\nu} &= \frac{1}{4}\delta((\xi P + q)^2)e_i^2 \mathrm{tr}\left[\xi \not{P}\gamma^\mu(\xi \not{P} + \not{q})\gamma^\nu\right] \\ &= \dots \\ &= \frac{\xi}{2}\delta(\xi - x)e_i^2\left[\left(-g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{2\xi}{P\cdot q}\left(P^\mu - q^\mu\frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu\frac{P\cdot q}{q^2}\right)\right]\end{aligned}$$

We see that the result is proportional to $\delta(1-\hat{x})$ with $\hat{x} = x/\xi$.

Structure functions in the parton model

- The contribution of a spin-1/2 parton of type i to the partonic tensor is given by:

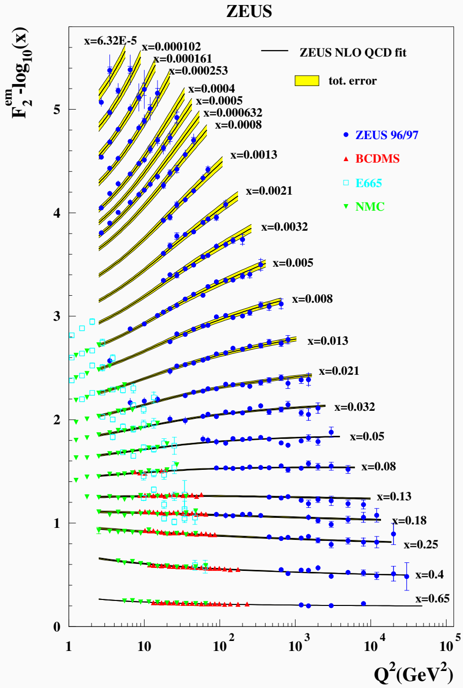
$$\hat{w}_i^{\mu\nu} = \frac{\xi}{2}\delta(\xi - x)e_i^2\left[\left(-g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{2\xi}{P\cdot q}\left(P^\mu - q^\mu\frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu\frac{P\cdot q}{q^2}\right)\right]$$

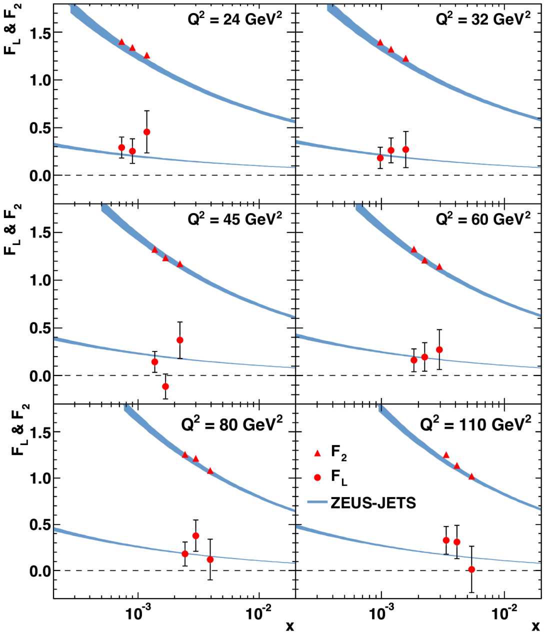
- If the parton density is given by $f_i(\xi)$, then the full contribution to the hadronic tensor reads:

$$W^{\mu\nu} = \sum_i \int_x^1 \frac{\mathrm{d}\xi}{\xi} f_i(\xi)\hat{w}_i^{\mu\nu}$$

- The corresponding structure functions read:

$$F_1(x, Q^2) = \frac{1}{2}\sum_i e_i^2 f_i(x),\ \ F_2(x, Q^2) = 2xF_1(x, Q^2),\ \ F_L(x, Q^2) = F_2(x, Q^2) - 2xF_1(x, Q^2) = 0$$


ZEUS
F_2^{em}-log$_{10}$(x)
Q^2(GeV^2)
ZEUS NLO QCD fit
tot. error
ZEUS 96/97
BCDMS
E665
NMC
x=6.32E-5
x=0.000102
x=0.000161
x=0.000253
x=0.0004
x=0.0005
x=0.000632
x=0.0008
x=0.0013
x=0.0021
x=0.0032
x=0.005
x=0.008
x=0.013
x=0.021
x=0.032
x=0.05
x=0.08
x=0.13
x=0.18
x=0.25
x=0.4
x=0.65



F_L & F_2
Q^2 = 24 GeV^2
Q^2 = 32 GeV^2
Q^2 = 45 GeV^2
Q^2 = 60 GeV^2
Q^2 = 80 GeV^2
Q^2 = 110 GeV^2
F_2
F_L
ZEUS-JETS
1.5
1.0
0.5
0.0
10^{-3}
10^{-2}
x

Structure functions in the parton model

The parton model can explain:

- Bjorken scaling:
  $F_i(x, Q^2) = F_i(x)$ for $Q^2 \to \infty$, $\nu \to \infty$ with $x = Q^2/2M\nu$ fixed
- The Callan-Gross relation:
  $F_2 = 2xF_1$ or $F_L = 0$, which holds for spin-1/2 fermions (for spin-0 we have i.e. $F_2 = F_L$).
- BUT:
  - How can the partons be free in a strongly bound state?

    asymptotic freedom
  - F_L is not 0.

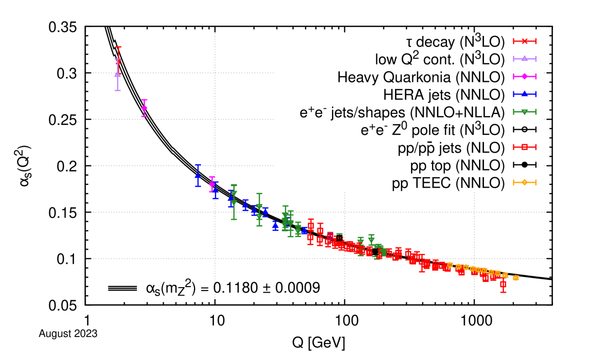
    higher order corrections, Z exchange, quark masses
  - What is the field theoretic description of the parton model?

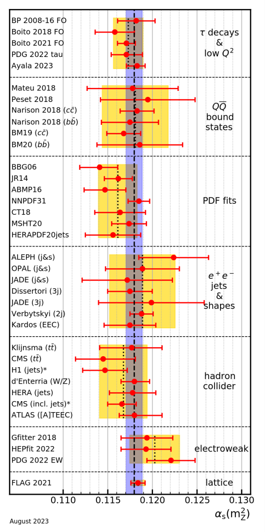
    operator product expansion

Asymptotic Freedom

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\alpha_s}{4\pi} = \beta(a_s) = -\left(\frac{\alpha_s}{4\pi}\right)^2 \left(b_0 + b_1\frac{\alpha_s}{4\pi} + b_2\left(\frac{\alpha_s}{4\pi}\right)^2 + b_3\left(\frac{\alpha_s}{4\pi}\right)^3 + b_4\left(\frac{\alpha_s}{4\pi}\right)^4 + \dots\right)$$

$$b_0 = \frac{11}{3}C_F - \frac{4}{3}n_f T_F, \ \dots$$





Operator Product Expansion

Operator Product Expansion

How does the parton picture emerge from a field theoretic point of view?

- The hadronic tensor is related to the imaginary part of the forward compton amplitude via the optical theorem:

$$W_{\mu\nu}(p,q) = \frac{1}{2\pi} \operatorname{Im} T_{\mu\nu}(p,q)$$

with

$$T_{\mu\nu}(p,q) = i \int \mathrm{d}^4 z e^{iq\cdot z} \langle P \,|\, T\left(J^\dagger_\mu(z) J_\nu(0)\right) |\, P \rangle$$

- In the Bjorken-limit the hadronic tensor is dominated by contributions near the light-cone:

$$z^2 \sim 0$$

- In this setting we can expand the product of currents:

$$T\left(J^\dagger_\mu(z) J_\nu(0)\right) \sim \sum_{i,\tau,n} c_{\tau,\mu,\nu}(z^2)^{i,\mu_1,\dots,\mu_n} O_{\mu_1,\dots,\mu_n}(0)^{i,\tau}$$

- The $O^{i,\tau}(0)$ are different local operators with the same twist $\tau = dim - spin$ (spin n $\leftrightarrow$ symmetric traceless tensors with n indices).
- The c^i_τ are the Wilson coefficients, which scale like $c^i_\tau \sim \sqrt{z^2}^\tau$.
- The leading term in the expansion has twist $\tau = 2$, with the operators:

$$O^{\mathrm{NS}}_{q,r;\mu_1,\ldots,\mu_N} = i^{N-1}\mathcal{S}\left[\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms} ,$$

$$O^{\mathrm{S}}_{q,r;\mu_1,\ldots,\mu_N} = i^{N-1}\mathcal{S}\left[\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{trace terms} ,$$

$$O^{\mathrm{S}}_{g,r;\mu_1,\ldots,\mu_N} = 2i^{N-2}\mathcal{S}\left[F^a_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_N}F^{\alpha,a}_{\mu_N}\right] - \text{trace terms}$$

Operator Product Expansion

- We find for the forward compton amplitude:

$$T_{\mu\nu} = \langle P| \sum_{N,j} \left(\frac{1}{Q^2}\right)^N \Bigg[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) q_{\mu_1} q_{\mu_2} C^N_{L,j}$$

$$- \left(g_{\mu\mu_1} g_{\nu\mu_2} q^2 - g_{\mu\mu_1} q_\nu q_{\mu_2} - g_{\nu\mu_2} q_\mu q_{\mu_1} + g_{\mu\nu} q_{\mu_1} q_{\mu_2}\right) C^N_{2,j} \Bigg] q_{\mu_3} \ldots q_{\mu_N} O^{j,\{\mu_1,\ldots,\mu_N\}}$$

$$+ \text{higher twist} \Big| P \rangle$$

- The hadronic operator matrix elements are defined by:

$$\langle P \Big| O^{j,\{\mu_1,\ldots,\mu_N\}} \Big| P \rangle = p^{\{\mu_1} \ldots p^{\mu_N\}} A^j_{P,N}$$

$$T_{\mu\nu} = \sum_{N,j} \left(\frac{1}{x}\right)^N \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) C^N_{L,j} + \left(-g_{\mu\nu} - \frac{4x^2}{q^2} p_\mu p_\nu - \frac{2x}{q^2}(p_\mu q_\nu + p_\nu q_\mu)\right) d_{\mu\nu} C^N_{2,j} \right] A^j_{P,N}$$

Operator Product Expansion

$$T_{\mu\nu} = \sum_{N,j} \left(\frac{1}{x}\right)^N \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) C_{L,j}^N + \left(-g_{\mu\nu} - \frac{4x^2}{q^2} p_\mu p_\nu - \frac{2x}{q^2}(p_\mu q_\nu + p_\nu q_\mu)\right) d_{\mu\nu} C_{2,j}^N \right] A_{P,N}^j$$

- We find an expansion for unphysical x ($x \to \infty$), which defines Mellin moments.
- The hadronic matrix elements $A_{P,N}^j$ are related to (moments) of the parton densities.
- For the calculation of the perturbative Wilson coefficients we use partonic states.
  $\Rightarrow$ Then all loop corrections to the matrix elements vanish.

$$\langle i \big| O^{j,\{\mu_1,\ldots,\mu_N\}} \big| j \rangle \sim \delta_{i,j}, \qquad i,j = \mathrm{q,g}$$

Mellin-Space

- In the parton picture we saw:

$$W^{\mu\nu} \sim \int_x^1 \frac{\mathrm{d}\xi}{\xi} f_i(\xi)\hat{w}_i^{\mu\nu}(\xi) = \int_0^1 \mathrm{d}\, y_1 \int_0^1 \mathrm{d}\, y_2 f_i(y_1)\hat{w}_i^{\mu\nu}(y_2)\delta(x - y_1 y_2) = \left[f_i \otimes \hat{w}_i^{\mu\nu}\right](x)$$

- Mellin transformation:

$$\mathcal{M}\left[g\right](N) = \int \mathrm{d}x\, x^{N-1}\, g(x)$$

- The Mellin transform diagonalizes the convolution:

$$\mathcal{M}\left[f \otimes g\right](N) = \int \mathrm{d}x x^{N-1} \left[f \otimes g\right](x) = \mathcal{M}[f](N)\,\mathcal{M}[g](N)$$

- Tomorrow also the generating function will be important:

$$\mathcal{G}[f](t) = \sum_{N=1}^{\infty} t^N\, \mathcal{M}\left[f\right](N)$$

Mellin-Space – Relations between different spaces

$$\hat{f}(t) = \sum_{N=1}^{\infty} t^N \, \tilde{f}(N)$$

$$\updownarrow$$

$$\tilde{f}(N) = \int_0^1 \mathrm{d}x \, x^{N-1} \, f(x)$$

$$\updownarrow$$

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-N} \tilde{f}(N) \mathrm{d}N$$

- $\hat{f}(t) \to \tilde{f}(N)$ and $\hat{f}(x) \to \tilde{f}(N)$: calculable via recurrence equations
- $\tilde{f}(N) \to f(x)$: calculable via differential equations
- $\hat{f}(t) \to f(x)$: calculable via analytic continuation

but: algorithmic solution only possible if recurrences or differential equations factorize to first order

Mellin-Space – Relations between different spaces

$$\hat{f}(t) = \sum_{N=1}^{\infty} \tilde{f}(N) t^N = \sum_{N=1}^{\infty} \int_0^1 \mathrm{d}x' \; t^N {x'}^{N-1} f(x') = \int_0^1 \mathrm{d}x' \; \frac{t}{1 - tx'} f(x')$$

Setting $t = \frac{1}{x}$ we obtain:

$$\hat{f}\left(\frac{1}{x}\right) = \int_0^1 \mathrm{d}x' \frac{f(x')}{x - x'}$$

Mellin-Space – Relations between different spaces

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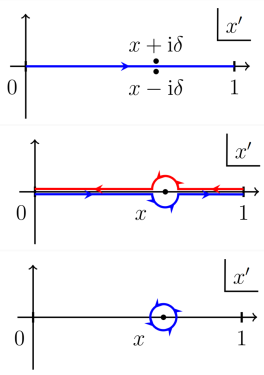


Setting $t = \frac{1}{x}$ we obtain:

$$\hat{f}\left(\frac{1}{x}\right) = \int_0^1 \mathrm{d}x' \frac{f(x')}{x - x'}$$

Therefore:

$$f(x) = \frac{i}{2\pi} \lim_{\delta \to 0} \oint_{|x - x'| = \delta} \frac{f(x')}{x - x'} = \frac{i}{2\pi} \operatorname*{Disc}_{x} \hat{f}\left(\frac{1}{x}\right)$$

Beyond Bjorken Scaling

- The Wilson coefficients above are not infrared safe.
- The operators need to be renormalized:

$$O^j = \sum_{q,g} Z_{jk} O^{k,ren} \quad , j = q, g$$

- The operator renormalization cancels the remaining infrared poles of the Wilson coefficients (mass factorization).

$$T^N_{F_i,p} = \left[C^N_{F_i,q} Z_{qp} + C^N_{F_i,g} Z_{gp} \right] A^{p,N}_{p,\mathrm{ren}}$$

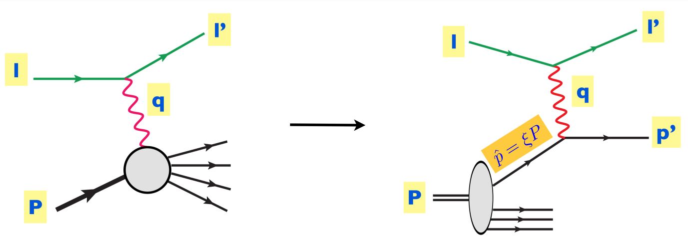
- We find a renormalization group equation:

$$\sum_{k=q,g} \left[\left\{ \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} + \beta \right\} \delta_{ik} - \gamma_{ik} \right] C_{i,k} = 0,$$

with the anomalous dimensions (splitting functions)

$$\gamma_{ij} = \left[\left(\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} Z \right) Z^{-1} \right]_{ij}$$

Applicability of the parton picture



- We saw two pictures of the proton:
  - the partonic picture of the proton at short distances
  - the operator picture of the proton at short distances
- Both pictures yield the same results at twist $\tau = 2$.
- Only the latter one can be extended consistently.

Applicability of the parton picture

- The parton picture is valid if the interaction time of the virtual gauge boson with the hadron τ_{int} is small compared to the life-time of individual partons $\tau_{\text{int}}/\tau_{\text{life}} \ll 1$.

- Approximately we have:

$$\tau_{\text{int}} \sim \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)} ,$$

$$\tau_{\text{life}} \sim \frac{1}{\sum_i (E_i - E)} = \frac{2P}{\sum_i (k_{\perp,i}^2 + m_i^2)/x_i - M^2} ,$$

- As a model: $m_i^2, M^2 = 0$, $k_{\perp,i}^2 = k_\perp^2$, $x_1 = x$, $x_2 = 1 - x$:

$$\frac{\tau_{\text{life}}}{\tau_{\text{int}}} = \frac{Q^2(1-x)^2}{2k_\perp^2} \gg 1$$

i.e. $Q^2 \gg k_\perp^2$ and x neither close to 1 or 0 ($xP \gg 1$).

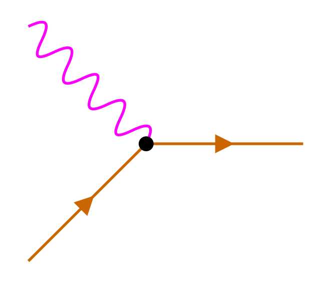
- There is no parton model at low Q^2, there also the light-cone expansion breaks down.

Heavy Quark Production in DIS

Heavy Quark Production in DIS

There are two possible ways for heavy quark production in DIS:

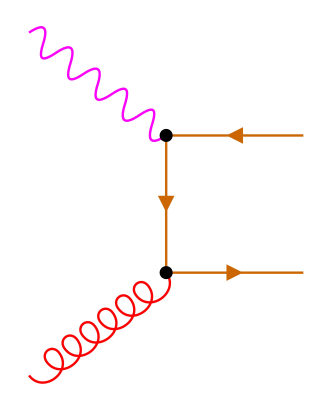
1. Intrinsic heavy quark contributions:
   - Postulate that there is a heavy quark component to the nucleon wave function.
   - Treat the heavy quark in the same way as the light quarks in the factorization of the structure functions.
   - Several experimental and theoretical studies suggest that the intrinsic contribution is small.
   - We will not discuss these contributions further.



Heavy Quark Production in DIS

There are two possible ways for heavy quark production in DIS:

2. Extrinsic heavy quark production:
   - The heavy quarks are only produced in the final state (or as virtual states).
   - The proton wave unction only contains massless quarks (u,d,s) and gluons (g).
     ⇒ This scheme is also known as the fixed flavor number scheme (FFNS).



Heavy Quark Production at LO

Consider $F_2^{c\bar{c}}(x, Q^2)$:

$$F_2^{\mathrm{LO}}(x, Q^2) = e_Q^2 a_s(Q^2) \int_{ax}^{1} \frac{\mathrm{d}y}{y} H_{2,g}^{(1)}\left(x/y, \frac{m_c^2}{Q^2}\right) G(y, Q^2)$$

$$\begin{aligned} H_{2,g}^{(1)}\left(z, \frac{m_c^2}{Q^2}\right) &= 8T_F\Big\{\beta\left[-\frac{1}{2} + 4z(1-z) + 2\frac{m_Q^2}{Q^2}z(z-1)\right] + \Big[-\frac{1}{2} + z(1-z) \\ &\quad + 2\frac{m_c^2}{Q^2}z(3z-1) + 4\frac{m_c^4}{Q^4}z^2\Big]\ln\left(\frac{1-\beta}{1+\beta}\right)\Big\} \\ &\stackrel{Q^2 \gg m_c^2}{=} 4T_F\left[\left(z^2 + (1-z)^2\right)\ln\left(\frac{Q^2}{m_c^2}\frac{1-z}{z}\right) + 8z(1-z) - 1\right] \end{aligned}$$

Massless Wilson coefficient:

$$C_{2,g}^{(1)}(z, \frac{Q^2}{\mu^2}) = 4T_F\left[\left(z^2 + (1-z)^2\right)\ln\left(\frac{Q^2}{\mu^2}\frac{1-z}{z}\right) + 8z(1-z) - 1\right]$$

We see:

$$H_{2,g}^{(1)}(z, \frac{m_c^2}{Q^2}) \stackrel{Q^2 \gg m_c^2}{=} C_{2,g}^{(1)}(z, \frac{Q^2}{\mu^2}) \underbrace{-4T_F\left(z^2 + (1-z)^2\right)\ln\left(\frac{m_c^2}{\mu^2}\right)}_{=A_{Qg}^{(1)}}$$

Is this accidental?

Heavy Quark Production in the Asymptotic Limit

The structure functions factorize

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
We can split the Wilson coefficients into massless and heavy flavor contributions:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right) + \mathcal{O}\left(\frac{m^2}{Q^2}\right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs).

Heavy Quark Production in the Asymptotic Limit

- The partonic operator matrix elements are defined as

$$A_{ki} = \langle i \,| O_k |\, i \rangle$$

with the same twist $\tau = 2$ operators as before:

$$O^{\rm NS}_{q,r;\mu_1,\dots,\mu_N} = i^{N-1} \mathcal{S}\left[\bar{\psi}\gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2}\psi\right] - \text{trace terms} \,,$$
$$O^{\rm S}_{q,r;\mu_1,\dots,\mu_N} = i^{N-1} \mathcal{S}\left[\bar{\psi}\gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi\right] - \text{trace terms} \,,$$
$$O^{\rm S}_{g,r;\mu_1,\dots,\mu_N} = 2 i^{N-2} \mathcal{S}\left[F^{a}_{\mu_1 \alpha} D_{\mu_2} \dots D_{\mu_N} F^{\alpha,a}_{\mu_N}\right] - \text{trace terms}$$

- Since the heavy quark provides a scale, these quantities do not vanish beyond the tree-level.

The heavy flavor Wilson coefficients in the asymptotic limit:

$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),{\rm NS}}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),{\rm NS}}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),{\rm NS}}(N_F+1)\delta_2 + A_{qq,Q}^{(2),{\rm NS}}(N_F+1)C_{q,(2,L)}^{(1),{\rm NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),{\rm NS}}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),{\rm PS}}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),{\rm NS}}(N_F)\tilde{C}_{g,(2,L)}^{(1),{\rm NS}}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),{\rm PS}}(N_F)\right]$$

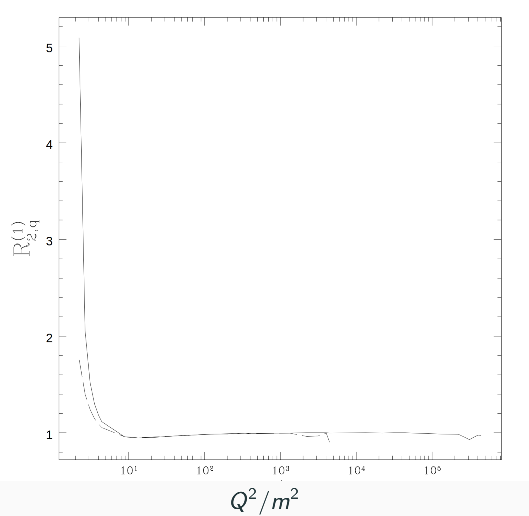
$$L_{g,(2,L)}^{\rm S}(N_F+1) = a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.$$
$$\left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),{\rm PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]$$

$$H_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^2\left[A_{Qq}^{(2),{\rm PS}}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),{\rm PS}}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qq}^{(3),{\rm PS}}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),{\rm PS}}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),{\rm NS}}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),{\rm PS}}(N_F+1)\right]$$

$$H_{g,(2,L)}^{\rm S}(N_F+1) = a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]$$
$$+ a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.$$
$$\left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),{\rm S}}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]$$

Validity of the Asymptotic Limit

- Comparison to exact $\mathcal{O}(\alpha_s^2)$:
  - $F_2^{c\bar{c}}$ needs $Q^2/m^2 \geq 10$
  - $F_L^{c\bar{c}}$ needs $Q^2/m^2 \geq 1000$



Comparison of the asymptotic and exact two loop contributions.

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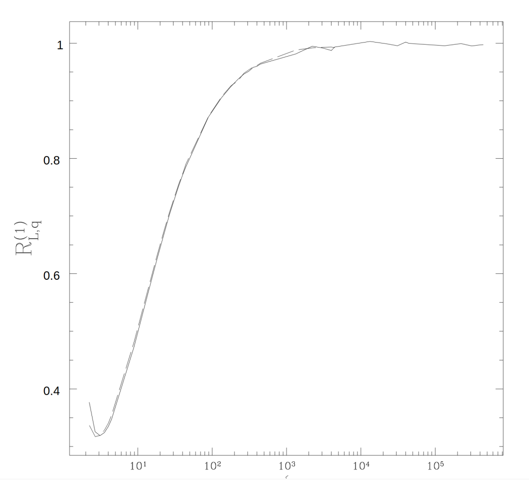
- Drawbacks:
  - Power corrections $(m^2/Q^2)^k$, $k \geq 1$ cannot be calculated.
  - Only inclusive quantities can be calculated.

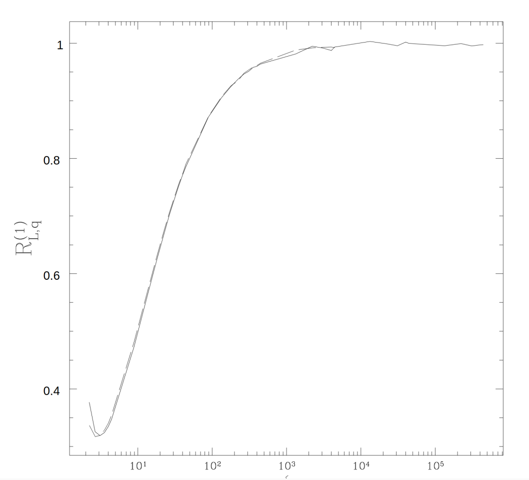


Q^2/m^2

Comparison of the asymptotic and exact two loop contributions.

Validity of the Asymptotic Limit

- Comparison to exact $\mathcal{O}(\alpha_s^2)$:
  - $F_2^{c\bar{c}}$ needs $Q^2/m^2 \geq 10$
  - $F_L^{c\bar{c}}$ needs $Q^2/m^2 \geq 1000$
- Advatanges:
  - Calculation much easier compared to full mass dependence.
    $\Rightarrow$ Opens up the possibility to consider two heavy quarks.
  - The massive OMEs can be used to define a variable flavor number scheme.



Q^2/m^2

Comparison of the asymptotic and exact two loop contributions.

Variable Flavor Number Scheme

- Idea: When $Q^2 \gg m^2$ we can treat the heavy quark effectively as massless.
- Demand for the structure functions:

$$F_i(n_f, Q^2) + F_i^{c\bar{c},asymp}(n_f, Q^2, m^2) = F_i^{VFNS}(n_f + 1, Q^2)$$

- By comparing both sides of the equation we can define new parton densities, which become dependent on the heavy quark mass.

Matching conditions for parton distribution functions:

$$f_k(N_F+1)+f_{\overline{k}}(N_F+1)=A_{qq,Q}^{\rm NS}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{\rm PS}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot G(N_F)\,,$$
$$f_Q(N_F+1)+f_{\overline{Q}}(N_F+1)=A_{Qq}^{\rm PS}\left(N_F+1,\frac{m_1^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot G(N_F)\,,$$
$$\Sigma(N_F+1)=\left[A_{qq,Q}^{\rm NS}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)+A_{qq,Q}^{\rm PS}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)+A_{Qq}^{\rm PS}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)+A_{Qg}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\right]\cdot G(N_F)\,,$$
$$G(N_F+1)=A_{gq,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot G(N_F)\,.$$

Variable Flavor Number Scheme

FFNS

- Fixed order in perturbation theory and fixed number of light partons in the proton.
- The heavy quarks are produced extrinsically only.
- The large logarithmic terms in the heavy quark coefficient functions entirely determine the charm component of the structure function for large values of Q^2.

VFNS

- Define a threshold above which the heavy quark is treated as light, thereby obtaining a parton density.
- Absorb mass singular terms from the asymptotic heavy quark coefficient functions and absorb them into parton densities.
- Resum large logarithms involving the mass.
- Provide heavy flavor initial state parton desities for the LHC, e.g. for $c\bar{s} \to W^+$.

Important:

- The VFNS is derived from the FFNS directly.
- New parton densities for the heavy quarks appear, which are now treated as light.
- Only universal (not power-supressed) terms are absorbed into the parton densities.

Summary and Outlook

Summary

Not discussed:

- Charged Current DIS
- Polarized DIS
- Target Mass Corrections
- Parton Densities
- Semi-Inclusive DIS
- ...

Outlook

Tomorrow:

- Higher order corrections to the massless, inclusive Wilson coefficients.
- Higher order corrections to the massive, inclusive Wilson coefficients.
- Methods for the calculation of massive operator matrix elements with one and two masses.

Backup

Inverse Mellin transform via analytic continuation

The discussion before used some implicit assumptions.
The x-space representation

1. has no $(-1)^N$ term.
2. is regular and has now contributions from distributions.
3. has a support only on $x \in (0, 1)$.

Inverse Mellin transform via analytic continuation

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For physical examples:

$$\tilde{f}(N) = \int_0^1 \mathrm{d}x \, x^{N-1} \left[f(x) + (-1)^N g(x) + \left(f_\delta + (-1)^N g_\delta \right) \delta(1-x) \right] + \int_0^1 \mathrm{d}x \, \frac{x^{N-1} - 1}{1-x}, \left[f_+(x) + (-1)^N g_+(x) \right]$$

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For physical examples:

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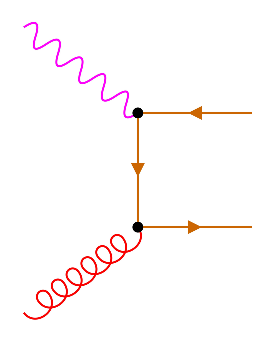
All of this can be lifted, but the discussion is more involved.

Massive Wilson Coefficients



We have to calculate the process:

$$q(p) + \gamma^*(q) \to Q(k_1) + Q(k_2),$$

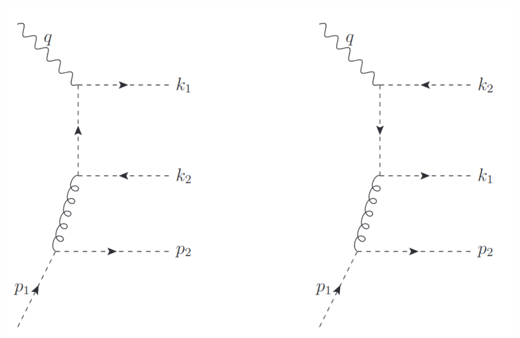
with $q^2 = -Q^2$, $p^2 = 0$, $k_1^2 = k_2^2 = m^2$, $(p+q)^2 = s = Q^2(1-z)/z$, $\beta = \sqrt{1 - 4m^2/s}$

- Parametrize the phase space:
  $p = \frac{s-Q^2}{2\sqrt{s}}(1,0,0,1)$, $k_1 = \frac{\sqrt{s}}{2}(1,0,\beta\cos\theta,\beta\sin(\theta))$, $k_2 = \frac{\sqrt{s}}{2}(1,0,-\beta\cos\theta,-\beta\sin(\theta))$

$$\int \mathrm{d}PS_2 = \int \frac{\mathrm{d}^4 k_1}{(2\pi)^d} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^d} (2\pi)^d \delta^{(d)}(p+q-k_1-k_2)(2\pi)^2 \delta(k_1^2)\delta(k_2^2)$$

$$= 2^{4-2d} \frac{\pi^{1-d/2}}{\Gamma(d/2-1)} s^{d/2-2} \beta^{d-3} \int_0^\pi \mathrm{d}\theta \sin^{d-3}(\theta)$$

Massive Wilson Coefficients – Pure-Singlet



$$\int d\,\mathrm{PS}_3 = \frac{1}{(4\pi)^d}\frac{(s-q^2)^{3-d}}{\Gamma(d-3)}\int_{s_{12}^-}^{s_{12}^+} ds_{12}\int_{t^-}^{t^+} dt\int_0^\pi d\theta\int_0^\pi d\phi\,[\sin(\theta)]^{d-3}\,[\sin(\phi)]^{d-4}$$
$$\times\; s_{12}^{d/2-2}\left[1-\frac{4m^2}{s_{12}}\right]^{d/2-3/2}\left[(s-q^2)u-q^2t\right]^{d/2-2}t^{d/2-2},$$

$$s_{12}^- = 4m^2, \quad s_{12}^+ = s,$$
$$t^- = 0, \quad t^+ = \frac{1}{s}(s-q^2)(s-s_{12}).$$

$$k_1 = \left(k^0, 0, \ldots, |\vec{k}|\sin(\phi)\sin(\theta), |\vec{k}|\cos(\phi)\sin(\theta), |\vec{k}|\cos(\theta)\right),$$
$$k_2 = \left(k^0, 0, \ldots, -|\vec{k}|\sin(\phi)\sin(\theta), -|\vec{k}|\cos(\phi)\sin(\theta), -|\vec{k}|\cos(\theta)\right),$$
$$p_1 = \frac{s-t-q^2}{2\sqrt{s_{12}}}\,(1,\ldots,0,0,1),$$
$$p_2 = \frac{s-s_{12}}{2\sqrt{s_{12}}}\,(1,0,\ldots,\sin(\chi),\cos(\chi)),$$