Series expansion of hypergeometric functions about their parameters using `MultiHypExp`

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and
Comput.Phys.Commun. 297 (2024) 109060 (arXiv:2306.11718)

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Advanced School & Workshop on Multiloop Scattering Amplitudes
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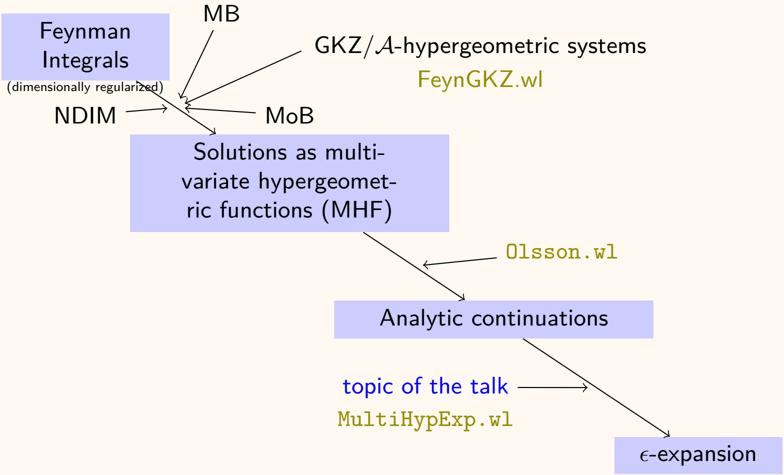
Outline

The Big Picture



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&= x(x+1)\ldots(x+n-1), \qquad x \in \mathbb{C} \setminus \mathbb{Z}_0^-, n \in \mathbb{Z}_0^+ \\
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- General hypergeometric functions

$$
{}_pF_{p-1}\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n \cdots \left(a_p\right)_n}{\left(b_1\right)_n \cdots \left(b_{p-1}\right)_n} \frac{z^n}{n!}, \qquad |z| < 1
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- **Horn Functions**: G_1, G_2, G_3 and $H_1, \ldots, H_7$
- **Lauricella functions**: F_A, F_B, F_C and F_D

$$F_C^{(3)} = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\left(a_1\right)_{n_1+n_2+n_3}\left(a_2\right)_{n_1+n_2+n_3}}{\left(c_1\right)_{n_1}\left(c_2\right)_{n_3}\left(c_3\right)_{n_2}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1!n_2!n_3!}$$

with domain of convergence : $\sqrt{|z_1|} + \sqrt{|z_2|} + \sqrt{|z_3|} < 1$

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- The sunset integral with unequal masses : Berends et. al. [2]

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- ϵ-expansion of the multivariate hypergeometric functions (MHF) are needed

From the literature

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  - Series
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- Available packages:
  - **Analytical** :
    - `HypExp` , `HypExp2` (T. Huber et. al. [3, 4])
    - `XSummer` (S. Moch et. al. [5])
    - `nestedsums` (S. Weinzierl [6, 7])
    - RISC packages : `Sigma` (C. Schneider [8]) , `EvaluateMultiSums` (C. Schneider [9]) , `HarmonicSums` (J. Ablinger [10])
  - **Numerical** : `NumExp` (Z.W. Huang et. al. [11])

Example 0

- Case I :

$$ {}_2F_1(1,1;\epsilon+1;x) = \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(\epsilon+1)_m}\frac{x^m}{m!} = \sum_{m=0}^{\infty} x^m + O(\epsilon) = \frac{1}{1-x} + O(\epsilon) $$

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- Proposed in [12]

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- MHF with *singular* parameters may have Laurent series expansion

Algorithm

- *Step 1*: Check if the Pochhammer parameters of the given function ($F(\epsilon)$) are *singular* or not
- *Step 2*: If those are non-singular, find the Taylor expansion of $F(\epsilon)$
- *Step 3*: If any of the Pochhammer parameter of $F(\epsilon)$ is *singular* then find a new function ($G(\epsilon)$) by replacing

  singular Pochhammer $\longrightarrow$ non-*singular* Pochhammer

- *Step 4*: Relate them using a differential operator (H)

$$F(\epsilon) = H(\epsilon) \bullet G(\epsilon)$$
$$\left[\sum_{i=-n}^{\infty} \epsilon^i H_i\right] \bullet \left[\sum_{j=0}^{\infty} \epsilon^j G_j\right]$$

Step 1 & 3 : Checking the Pochhammers

- *Singular* Pochhammers :

  1. When one or more lower Pochhammer parameters (i.e., Pochhammer parameters in the denominator) are of the form

  $$(-p + q\epsilon)_p$$

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- The Gauss ${}_2F_1$ example

$$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m} \frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m} \frac{x^m}{m!}$$

singular Pochhammer in A_1 : $(-1+\epsilon)_m$
non-*singular* Pochhammer in A_2 : $(1+\epsilon)_m$

Step 2 :Taylor Expansion

Obtaining DE

- From the denition of Taylor expansion

$$F(\epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \frac{d^i}{d\epsilon^i} F(\epsilon)\Big|_{\epsilon=0}$$

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$$\frac{A_{n+1}}{A_n} = \frac{(a+n)(b+n)}{(n+1)(c+n)} = \frac{g(n)}{h(n)}$$

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- The annihilator

$$L = \left[h(\theta) \frac{1}{x} - g(\theta) \right]$$

where $\theta = x\partial_x$

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Pfaff System

- For Gauss ${}_2F_1$: $L \bullet \ {}_2F_1(a, b; c; x) = 0$

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- Consider $g = ({}_2F_1, x\partial_x \bullet {}_2F_1)^T$

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- The length of the vector g = Holonomic rank of the system
- Find a transformation T to bring the system into *canonical* form (J. Henn [13])

$$dg' = \epsilon\Omega' g'$$

with

$$g = Tg' \qquad , \qquad \Omega' = T^{-1}\Omega T - T^{-1}dT$$

Step 2 :Taylor Expansion

Example

- For our example of $A_2 = {}_2F_1(\epsilon, -\epsilon; 1+\epsilon; x)$

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- **Boundary Condition :** At $x = 0$

$$g = (1, 0)^T$$

- **Solution :**

$$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$

Step 4 : Differential operator

Contiguous relations

There exist contiguous relations that relate

$$ {}_2F_1(a \pm 1, b; c; x) \ , \ {}_2F_1(a, b \pm 1; c; x) \ , \ {}_2F_1(a, b; c \pm 1; x) $$

These can be obtained by applying differential operators

- *Example*:
  The unit step down operator for the Gauss hypergeometric series is given by $H(c) = \frac{1}{c}(\theta + c)$, i.e.,

$$ {}_2F_1(a, b; c; x) = H(c) \bullet \ {}_2F_1(a, b; c + 1; x) $$

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- Another example

$$ {}_2F_1(a+1, b; c; x) = \frac{1}{a}(\theta + a) \bullet \ {}_2F_1(a, b; c; x) $$

Step 4 : Differential operator

Step Down Operators

- If needed, apply the step down operator multiple times

$$ {}_2F_1(a, b; c-1; x) = H(c-1)H(c) \bullet \; {}_2F_1(a, b; c+1; x) $$

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- Take quotient by the annihilator L

$$ L \bullet \ {}_2F_1(a, b; c+1, x) = 0 $$
$$ L = \left[-ab + (-x(a+b+1) + c + 1)\,\partial_x - (x-1)x\partial_x^2\right] $$

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- The step down operator :

$$ \begin{aligned} H &= H(c-1)H(c) \\ &= \left(1 - \frac{abx}{(c-1)c(x-1)}\right) - \frac{x(a+b+1)-2cx+c-1}{(c-1)c(x-1)}\theta \end{aligned} $$

Step 4 : Differential operator

Example

- $$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m}\frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m}\frac{x^m}{m!}$$

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- So $A_1 = H \bullet A_2$

$$\begin{aligned} H &= \frac{(\epsilon(2x-1)-x+1)}{(\epsilon-1)\epsilon(x-1)}\theta + \frac{\epsilon(2x-1)-x+1}{(\epsilon-1)(x-1)} \\ &= \frac{1}{\epsilon}\theta + \left(1 - \frac{x}{x-1}\theta\right) + O(\epsilon) \end{aligned}$$

Step 4 : Differential operator

Example

- $$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m} \frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m} \frac{x^m}{m!}$$
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- $$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$

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- $$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$
- $$A_1 = 1 + \epsilon\left[G(1;x) - \frac{x}{x-1}\right] + \epsilon^2\left[-\frac{x}{x-1}G(1;x) + G(1,1;x) - \frac{x}{x-1}\right] + O(\epsilon^3)$$

Mathematica Package

MultiHypExp

MultiHypExp

Available at GitHub [14]

Dependencies :

- RISC‘HolonomicFunctions (Koutschan [15, 16]) : To find the PDE associated with the given MHF and to form the Pfaff system
- HYPERDIRE (Bytev et.al. [17, 18, 19]) : For step up/down operations
- CANONICA (Meyer [20]) : To bring the Pfaff system into canonical form
- PolyLogTools (Duhr et. al. [21]) : To handle MPLs

Mathematica Package

MultiHypExp

The package is able to expand the following series

- **One variable** : ${}_pF_{p-1}$
- **Two variables** : Appell F_1, F_2, F_3, F_4, Horn $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_6$ and H_7 and certain KdF functions
- **Three variables** : Lauricella Saran $F_A, F_B, F_D, F_K, F_M, F_N$ and F_S
- Apart from Appell F_1, F_2, F_3 and Horn H_2, other Appell-Horn series are expanded using their relation to the former functions
- Series expansion of Appell F_4 and Horn H_1 is possible with certain restriction on the Pochhammer parameters

MultiHypExp

Commands for one variable

To obtain the series expansion ${}_2F_1(\epsilon, -\epsilon; \epsilon - 1; x)$

```
In[1]:=  SeriesExpand[{{e, -e}, {e - 1}}, {x}, e, 3]
Out[1]=
    1 + (-(x/(-1+x))+G[1,x]) e + (-(x/(-1+x))-(x G[1,x])
    /(-1+x)+G[1,1,x]) e^2 + O[e]^3
```

Alternatively,

```
In[2]:= SeriesExpand[{n},(Pochhammer[e,n]Pochhammer[-e,n]x^n)
    /(Pochhammer[e-1,n]n!),{x},e,3]
```

yields the same result.

MultiHypExp

Commands for bi- and tri-variate HF

In[3]:= `SeriesExpand[F2,{1,1,e,1+e,1-e},{x,y},e,3]`

Out[3]=

```
 -(1/(-1+x))+((-2 G[1,x]+G[1,y]+G[1-y,x]) e)/(-1+x)
+(1/(-1+x))(2 G[1,x] G[1,y]-2 G[1,y] G[1-y,x]
+2 G[0,1,x]+G[0,1,y]-G[0,1-y,x]-4 G[1,1,x]-2 G[1,1,y]
+2 G[1,1-y,x]+2 G[1-y,1,x]-G[1-y,1-y,x]) e^2+O[e]^3
```

yields the first three series expansion coefficients of Appell $F_2(1, 1, e; 1 + e, 1 - e; x, y)$ with respect to e in terms of MPLs.

In[4]:= `SeriesExpand[{m,n}, exp, {x, y}, e, 3]`

exp must be a series presentation of a MHF with summation indices m and n.

MultiHypExp

Commands for obtaining reduction formulae

To find reduction formulae of MHF

In[5]:=
```
ReduceFunction[F2,{3,2,1,3,2},{x,y}]
```

Out[5]=
```
1/((-1+x) x (-1+x+y))-G[1,x]/(x^2 y)+G[1-y,x]/(x^2 y)
```

In terms of logarithms

$$F_2(3,2,1;3,2;x,y) = -\frac{\log(1-x)}{x^2 y} + \frac{\log\left(1-\frac{x}{1-y}\right)}{x^2 y} + \frac{1}{(x-1)x(x+y-1)}$$

This command can find reduction formulae of Appell F_1, F_2, F_3, F_4 and Lauricella-Saran $F_D^{(3)}$ and $F_S^{(3)}$

MultiHypExp

Conclusions & Future works

- Applicable when the parameter ϵ appears linearly inside the Pochhammer symbols
- The package can find the expansion of most of the MHF around integer values of Pochhammer parameters
- It can find at most first 6 coefficients
- Not valid at singular locus of the function

Future works : Expansion around half-odd integer parameters

- Some change of variables required but series coefficients can be expressed in terms of MPLs (M.A. Bezuglov et. al [22])
- May require functions beyond MPLs

Thank You

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