Exploring GKZ hypergeometric systems for Feynman integral calculus

Based on: 2211.01285 [hep-th]

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Background 1: Typical pQCD workflow for precision studies

- $\mathcal{L}$, and some process containing elementary particles (the hard interaction). Computation at some fixed-order.
- Generate all the diagrams - `QGRAF, FeynArts`, etc.
- Perform the color and Lorentz algebra to extract the scalar part of the diagrammatic amplitudes - `color, FeynCalc, tapir`, etc.

Background 1: Typical pQCD workflow for precision studies (contd.)

- Choose an optimal set of topologies (or integral-families) - `q2e/exp`, `tapir`, `FeynCalc`, etc.
- Perform reduction to a set of master integrals for a set of seed integrals in each topology - `KIRA`, `FIRE`, `Reduze`, `LiteRed`, etc.
- Solve these master integrals using the method of differential equations. A ”good”-choice for the basis of master integrals can simplify solving the system of differential equations - `CANONICA`, `epsilon`, `fuchsia`, `INITIAL`, `Libra`, etc.

Background 1: Typical pQCD workflow for precision studies (contd.)

- Solving the system of differential equations requires a knowledge of the relevant boundary-conditions. Often, an analytic result is preferred.
- The Mellin-Barnes (MB) method is handy for performing computations of individual scalar Feynman integrals - `AMBRE`, `MB`, `MBresolve`, etc.

Background 1: Typical pQCD workflow for precision studies (contd.)

- Limitation of the MB approach: not well-suited for handling integrals with a large number of scales. Complexity is reflected in the number of Mellin-Barnes variables required to represent the given Feynman integral as an MB integral $\equiv$ in the summation-fold of the nested-sums that such MB integrals could be converted to through residue computation - `MBConicHulls`[1].

[1] Ananthanarayan et al. 2021a.

Background 2: ”Hypergeometrics” in Feynman integral calculus

- Feynman integrals as a set of ”generalized hypergeometric functions”. Singularities of these functions coincide with the Landau singularities[2].
- Taking the sums of residues in the MB approach yields several functions of the ”hypergeometric” type - Appell, Lauricella, Lauricella-Saran, etc
- A given Feynman integral can be represented by ”hypergeometric” integrals, such as the Meijer G-function, or the Fox H-function.

[2] Kashiwara et al. 1977; Regge 1968.

Background 3: Gel'fand, Graev, Kapranov, Zelevinsky

- Systematic and consistent generalization of the concept of "hypergeometric" functions.
- The G(G)KZ approach can be used to solve and study classes of integrals, such as Euler integrals[3].
- First known contact with physics: arxiv.9308122[4], arxiv.9406055[5].

[3] I. Gelfand et al. 1987.
[4] Hosono et al. 1995a.
[5] Hosono et al. 1995b.

Idea and Questions

- Explore the scope of the GKZ approach in analysing and evaluating individual scalar Feynman integrals.
- Possible to bypass the MB-representation? Or at least the (multivariate) residue computation step that is typical of the MB-approach?
- What do the solutions look like?
- Most importantly, does bypassing the residue computation necessarily indicate a better algorithm? What are the limitations of the GKZ approach, when compared against the MB approach, as automated in the `MBConicHulls` package?

Our contribution

A proof-of-concept implementation demonstrating the utility of the GKZ approach in evaluating individual scalar Feynman integrals in the form of a *Mathematica* package `FeynGKZ`[6].

[6] Ananthanarayan et al. 2022.

Table of contents

Feynman integrals

The momentum representation

- Typically involve tensor structures in numerator - do tensor reduction
- Calculate the scalar integrals
- Momentum representation:

$$I_\Gamma(\nu, D) = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{n}(-q_j^2 + m_j^2)^{\nu_j}} \tag{1}$$

l: number of loops
D: the space-time dimension
$\nu = (\nu_1, ..., \nu_n)$: propagator powers
k_r-s and q_j-s are the loop-momenta and internal-momenta for the Feynman graph Γ, respectively.

The Lee-Pomeransky representation

- An alternate form[7]:

$$
\begin{aligned}
I_\Gamma(\nu, D) &= \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2}-\omega)}\Big(\prod_{i=1}^{n}\int_{\alpha_i=0}^{\infty}\frac{d\alpha_i\,\alpha_i^{\nu_i-1}}{\Gamma(\nu_i)}\Big)G(\alpha)^{-\frac{d}{2}} \\
&= \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2}-\omega)\Gamma(\nu)}\int_{\mathbb{R}_+^n} d\alpha\,\alpha^{\nu-1}\,G(\alpha)^{-\frac{d}{2}}
\end{aligned}
\tag{2}
$$

- Lee-Pomeransky polynomial: $G(\alpha) = U(\alpha) + F(\alpha)$.

[7] Lee et al. 2013.

The Lee-Pomeransky representation (contd.)

- Generalized G-polynomial:

$$G_z(\alpha) = \sum_{a_j \in A} z_j \alpha^{a_j} = \sum_{j=1}^{N} z_j \prod_{i=1}^{n} \alpha_i^{a_{ij}} \tag{3}$$

$z_j \to$ generic/indeterminate

- Generalized Feynman integral:

$$I_{G_z}(\nu, \nu_0) = \Gamma(\nu_0) \int_{\mathbb{R}_+^n} d\alpha \, \alpha^{\nu - 1} G_z(\alpha)^{-\nu_0} \tag{4}$$

where, $\nu_0 = \frac{D}{2}$

The associated GKZ system and its solutions

The associated GKZ system

$I_{G_z}(\nu, \nu_0)$ satisfies a holonomic system of PDEs called a *GKZ hypergeometric system*[8].

Ideals
Let $P = \mathbb{F}[x_1, ..., x_n]$ be some polynomial ring in $x_1, ..., x_n$ over $\mathbb{F}$.
$\mathcal{I} \subset P$ is said to be an ideal if

- $0 \in \mathcal{I}$
- $f + g \in \mathcal{I} \quad \forall\, f, g \in \mathcal{I}$
- $f \cdot g \in \mathcal{I} \quad \forall\, f \in P, g \in \mathcal{I}$

Thus, $\langle \mathcal{S} \rangle = \sum_i f_i g_i;\ f \in P, g \in S$ is the ideal spanned by $S \subset P$.

[8] I. Gelfand et al. 1990, 1994.

The associated GKZ system (contd.)

- We describe the GKZ system as follows:

$$H_{\mathcal{A}}(\underline{\nu}) = I_{\mathcal{A}} \cup \langle \mathcal{A} \cdot \theta + \underline{\nu} \rangle \tag{5}$$

$$\begin{aligned} \mathcal{A} &= \{a_{ij}; i \in \{1, ..., n+1\}, j \in \{1, ..., N\}\} | a_{ij} = 1; i = 1\} \\ \underline{\nu} &= (\nu_0, \nu_1, ..., \nu_n)^T \end{aligned} \tag{6}$$

- $\mathcal{A} \rightarrow (n+1) \times N$ matrix; $n+1 \leq N$
- *Codimension* of $\mathcal{A}$: $N - n - 1$
- $\theta = (\theta_1, \ldots, \theta_N)^T$; $\theta_i = z_i \partial_i \rightarrow$ Euler operators
- **Assume**: (1,...,1) lies in $\mathbb{Q}$-row span of $\mathcal{A}$

The associated GKZ system (contd.)

- $H_{\mathcal{A}}(\underline{\nu}) I_{G_z}(\nu, \nu_0) = 0$
- $I_{G_z}(\nu, \nu_0) \rightarrow$ *GKZ hypergeometric function*![9]

[9] Cruz 2019; Klausen 2020.

Solving the GKZ system

- **Algebraically**: the SST algorithm[10] $\rightarrow$ the Gröbner deformation method.
- **Geometrically**: the triangulation method.
- Both are equivalent!
- Basically, there exists a bijective-map b/w what are called ”square-free initial ideals” and the ”unimodular regular triangulations”
- In this talk, focus on the **geometric** picture.

[10] Saito et al. 2013.

Solving the GKZ system (contd.)

- We saw:

$$\mathcal{A} = \begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{(n+1)\times N} \tag{7}$$

- A defines an assembly of N points (a point configuration) in $\mathbb{Z}^n$

$$\text{Conv}(A) := \left\{ \sum_{j=1}^{N} k_j a_j \middle| k \in \mathbb{R}^N_{\geq 0}, \sum_{j=1}^{N} k_j = 1 \right\} \tag{8}$$

- Newton polytope of $G_z(\alpha)$:

$$\Delta_{G_z} := Conv(A) \tag{9}$$

Solving the GKZ system (contd.)

- Triangulate Δ_{G_z}!
- Triangulation structure: $T = \{\sigma_1, ..., \sigma_r\}$.
- $\sigma_i \subset \{1, ..., N\}$ is some index set.

Solving the GKZ system (contd.)

Can always obtain a regular triangulation![11]

[11] I. M. Gelfand et al. 1991.

Solving the GKZ system (contd.)

Can always obtain a unimodular regular triangulation ($\text{vol}_0(\sigma_i) = 1$)![12]

[12] Bruns et al. n.d.; Knudsen 1973.

Solving the GKZ system (contd.)

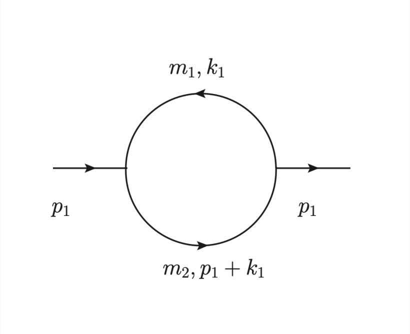
- Regular triangulations can be used to construct a basis for the finite-dimensional solution space of $H_{\mathcal{A}}(\underline{\nu})$.
- Each element: Γ-series, due to a string of Γ-functions appearing in both the numerator and the denominator. **Pingback** to one of our initial questions: *what do the solutions look like?*
- Whole solution: linear combination of the Γ-series elements.
- Unimodularity: one $\sigma_i \rightarrow$ one Γ-series.
- Might as well use just the unimodular regular triangulations to construct a basis!

Demonstration

Demonstration

Example 1: *Bubble diagram with two unequal masses*

Bubble diagram with two unequal masses



The corresponding integral in momentum-representation:

$$I_\Gamma(\nu_1, \nu_2, D; p_1^2) = \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_1^2 + m_1^2)^{\nu_1}(-(p_1 + k_1)^2 + m_2^2)^{\nu_2}}$$

with two unequal masses m_1 and m_2, and external momentum p_1.

Bubble diagram with two unequal masses (contd.)

After successfully loading the package and installing its dependencies, specify the integral in its momentum representation as:

```
In[3]:=    MomentumRep = {{k1, m1, a1}, {p1 + k1, m2, a2}};
           LoopMomenta = {k1};
           InvariantList = {p1^2 -> -s};
           Dim = 4 - 2ϵ;
           Prefactor = 1;
```

Bubble diagram with two unequal masses (contd.)

Now derive the $\mathcal{A}$-matrix:

```
In[4]:=   FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta,
           InvariantList, Dim, Prefactor}, UseMB → False];
```

Prints ⇒ The Symanzik polynomials → $U = x_1 + x_2$
$, F = m_1^2 x_1^2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$

The Lee-Pomeransky polynomial → $G =$
$x_1 + m_1^2 x_1^2 + x_2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$

The associated $\mathcal{A}$−matrix → $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$, which has codim = 2.

Normalized Volume of the associated Newton Polytope → 3

Time Taken 1.50005 seconds

Bubble diagram with two unequal masses (contd.)

Compute the unimodular regular triangulations[13]:

```
In[5]:=    Triangulations = FindTriangulations[FindAMatrixOut];
```

```
Prints ⇒   Finding all regular triangulations ...
           Found 5 Regular Triangulations, out of which 3 are Unimodular
           The 3 Unimodular Regular Triangulations →
           1 ::  {{1,2,3},{2,3,4},{3,4,5}}
           2 ::  {{1,2,3},{2,4,5},{2,3,5}}
           3 ::  {{2,4,5},{1,3,5},{1,2,5}}
           Time Taken 0.126965 seconds
```

[13] Rambau 2002.

Bubble diagram with two unequal masses (contd.)

Calculate the Γ-series:

```
In[7]:=   SeriesSolution = SeriesRepresentation[Triangulations,2];
```

Prints ⇒

```
Unimodular Triangulation → 2
Number of summation variables → 2
```

Non-generic limit → $\{z_1 \to m_1^2,\ z_2 \to s + m_1^2 + m_2^2,\ z_3 \to 1,\ z_4 \to m_2^2,\ z_5 \to 1\}$

The series solution is the sum of following 3 terms.

Term 1 ::

$$\left(\left((-1)^{-n_1-n_2}\,\text{Gamma}[-2+\epsilon+a_1-n_1-n_2]\,\text{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\text{Gamma}[a_2+2n_1+n_2]\,(m_1^2)^{2-\epsilon-a_1}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_2}\right)\Big/\left(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2]\right)\right)$$

Term 2 ::

$$\left(\left((-1)^{-n_1-n_2}\,\text{Gamma}[-2+\epsilon+a_2-n_1-n_2]\,\text{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\text{Gamma}[a_1+2n_1+n_2]\,(m_2^2)^{2-\epsilon-a_2}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_1}\right)\Big/\left(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2]\right)\right)$$

Term 3 ::

$$\left(\left((-1)^{-n_1-n_2}\text{Gamma}[2-\epsilon-a_2+n_1-n_2]\,\text{Gamma}[2-\epsilon-a_1-n_1+n_2]\,\text{Gamma}[-2+\epsilon+a_1+a_2+n_1+n_2]\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\right)\Big/\left(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2]\right)\right)$$

```
Time Taken 0.066558 seconds
```

Bubble diagram with two unequal masses (contd.)

Check for an expression in terms of known hypergeometric functions[14]:

In[8]:= `GetClosedForm[SeriesSolution];`

Prints ⇒ `Closed form found with Olsson!`

Term 1 ::

$$\frac{1}{\text{Gamma}[a_1]}\,\text{Gamma}[-2+\epsilon+a_1]\;\text{H3}\left[a_2, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_1, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_1^2}{s+m_1^2+m_2^2}\right]\; m_1^4\,(m_1^2)^{-\epsilon-a_1}(s+m_1^2+m_2^2)^{-a_2}$$

Term 2 ::

$$\frac{1}{\text{Gamma}[a_2]}\,\text{Gamma}[-2+\epsilon+a_2]\;\text{H3}\left[a_1, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_2, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_2^2}{s+m_1^2+m_2^2}\right]\; m_2^4\,(m_2^2)^{-\epsilon-a_2}(s+m_1^2+m_2^2)^{-a_1}$$

Term 3 ::

$$\left(\left(\text{G1}\left[-2+\epsilon+a_1+a_2, 2-\epsilon-a_1, 2-\epsilon-a_2, -\frac{m_2^2}{s+m_1^2+m_2^2}, -\frac{m_1^2}{s+m_1^2+m_2^2}\right]\,\text{Gamma}[2-\epsilon-a_1]\,\text{Gamma}[2-\epsilon-a_2]\;\text{Gamma}[-2+\epsilon+a_1+a_2]\,(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\right)\Big/\left(\text{Gamma}[a_1]\;\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\right)\right)$$

`Time Taken 0.05827 seconds`

---

[14] Ananthanarayan et al. 2021b.

Bubble diagram with two unequal masses (contd.)

Evaluate the sum of the Γ-series terms numerically:

```
In[9]:=   SumLim = 30;
          ParameterSub = {ϵ → 0.001, a1 → 1, a2 → 1, s → 10, m1 → 0.4, m2 → 0.3};
          NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

```
Prints ⇒  Numerical result = 997.382
          Time Taken 0.222572 seconds
```

Example 2: *Two-loop self-energy with four propagators*

Two-loop self-energy with four propagators

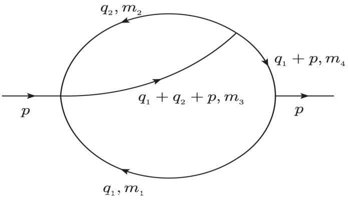


FIG. 1: The 2-loop self energy with 4 propagators.

The corresponding integral in momentum-representation:

$$I_\Gamma(\nu_1, \nu_2, \nu_3, \nu_4, D; p^2) = \int \frac{d^D q_1 \, d^D q_2}{(i\pi^{\frac{D}{2}})^2} \quad \times$$

$$\frac{1}{(-q_1^2 + m_1^2)^{\nu_1}(-q_2^2 + m_2^2)^{\nu_2}(-(q_1 + q_2 + p)^2 + m_3^2)^{\nu_3}(-(q_1 + p)^2 + m_4^2)^{\nu_4}}$$

with four unequal masses m_1, m_2, m_3 and m_4, and external momentum p.

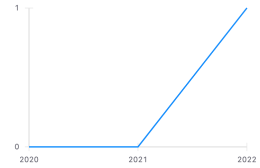
Two-loop self-energy with four propagators (contd.)

GKZ-system of the 2-loop self energy with 4 propagators

Tai-Fu Feng (Hebei U. and KLHCAQFT, Baoding and Guangxi U. and Chongqing U.), Hai-Bin Zhang (Hebei U. and KLHCAQFT, Baoding and Guangxi U.), Yan-Qing Dong (Hebei U. and KLHCAQFT, Baoding), Yang Zhou (Hebei U. and KLHCAQFT, Baoding)

Sep 29, 2022

299 pages
e-Print: 2209.15194 [hep-th]
View in: ADS Abstract Service



Abstract: (arXiv)
Applying the system of linear partial differential equations derived from Mellin-Barnes representations and Miller's transformation, we present GKZ-system of Feynman integral of the 2-loop self energy diagram with 4 propagators. The codimension of derived GKZ-system equals the number of independent dimensionless ratios among the external momentum squared and virtual mass squared. In total 536 hypergeometric functions are obtained in neighborhoods of origin and infinity, in which 30 linearly independent hypergeometric functions whose convergent regions have non-empty intersection constitute a fundamental solution system in a proper subset of the whole parameter space.

Note: latex, 299 pages, including 1 figure + 17 appendices. arXiv admin note: text overlap with arXiv:2206.04224

02.30.Jr 11.10.Gh 12.38.Bx propagator Feynman graph differential equations Mellin transformation

Two-loop self-energy with four propagators (contd.)

What we obtain from `FeynGKZ` for this integral:

- $\mathcal{A}$-matrix of codimension 4, thus, four summation variables.
- **Trick**: MB-representation informed $\mathcal{A}$-matrix[15], in contrast to the LP-representation based one that we considered earlier.
- Numerically verified against `FIESTA`, for a given kinematic point.

[15] Feng et al. 2020.

Two-loop self-energy with four propagators (contd.)

Pingbacks to one of our initial questions: *possible to bypass the MB-representation? Or at least the (multivariate) residue computation step that is typical of the MB-approach?*

- **Pingback** to our question about bypassing the MB representation: can be done in principle by using the LP representation instead, considering the MB representation often simplifies things a lot. Namely, we have the following identity:

$$\begin{aligned}&\text{no. of MB integration variables}\\&=\text{codimension of }\mathcal{A}-\text{matrix}\\&=\text{no. of }\Gamma\text{-series summation variables}\end{aligned}$$

- **Pingback** to our question about bypassing the multivariate residue computation step in the traditional MB approach: can be done in the GKZ framework by considering triangulations instead.

Two-loop self-energy with four propagators (contd.)

Derive the $\mathcal{A}$-matrix:

$\mathcal{A}$ - matrix

```
In[∘]:= FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta, InvariantList, Dim, Prefactor}, UseMB → True];
```

The Symanzik polynomials → $U = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_2 x_4 + x_3 x_4$, $F = m_1^2 x_1^2 x_2 + m_2^2 x_1 x_2^2 + m_1^2 x_1^2 x_3 + s\, x_1 x_2 x_3 + m_1^2 x_1 x_2 x_3 + m_2^2 x_1 x_2 x_3 + m_3^2 x_1 x_2 x_3 + m_2^2 x_2^2 x_3 + m_3^2 x_1 x_3^2 + m_3^2 x_2 x_3^2 + s\, x_1 x_2 x_4 + m_1^2 x_1 x_2 x_4 + m_4^2 x_1 x_2 x_4 + m_2^2 x_2^2 x_4 + s\, x_1 x_3 x_4 + m_1^2 x_1 x_3 x_4 + m_4^2 x_1 x_3 x_4 + m_2^2 x_2 x_3 x_4 + m_3^2 x_2 x_3 x_4 + m_4^2 x_2 x_3 x_4 + m_3^2 x_3^2 x_4 + m_4^2 x_2 x_4^2 + m_4^2 x_3 x_4^2$

The Lee-Pomeransky polynomial → $G = x_1 x_2 + m_1^2 x_1^2 x_2 + m_2^2 x_1 x_2^2 + x_1 x_3 + m_1^2 x_1^2 x_3 + x_2 x_3 + s\, x_1 x_2 x_3 + m_1^2 x_1 x_2 x_3 + m_2^2 x_1 x_2 x_3 + m_3^2 x_1 x_2 x_3 + m_2^2 x_2^2 x_3 + m_3^2 x_1 x_3^2 + m_3^2 x_2 x_3^2 + x_2 x_4 + s\, x_1 x_2 x_4 + m_1^2 x_1 x_2 x_4 + m_4^2 x_1 x_2 x_4 + m_2^2 x_2^2 x_4 + x_3 x_4 + s\, x_1 x_3 x_4 + m_1^2 x_1 x_3 x_4 + m_4^2 x_1 x_3 x_4 + m_2^2 x_2 x_3 x_4 + m_3^2 x_2 x_3 x_4 + m_4^2 x_2 x_3 x_4 + m_3^2 x_3^2 x_4 + m_4^2 x_2 x_4^2 + m_4^2 x_3 x_4^2$

The Mellin-Barnes representation →

$$\Big(\big(s^{4-z1-z2-z3-z5-2\epsilon-a_1-a_2-a_3-a_4}\,\mathrm{Gamma}[-z1]\,\mathrm{Gamma}[-z2]\,\mathrm{Gamma}[-z3]\,\mathrm{Gamma}[-z5]\,\mathrm{Gamma}[2-z3-\epsilon-a_1]\,\mathrm{Gamma}[2-z1-\epsilon-a_2]\,\mathrm{Gamma}[2-z2-\epsilon-a_3]\,\mathrm{Gamma}[-2+z1+z2+\epsilon+a_2+a_3]$$
$$\mathrm{Gamma}[4-z1-z2-z5-2\epsilon-a_2-a_3-a_4]\,\mathrm{Gamma}[z5+a_4]\,\mathrm{Gamma}[-4+z1+z2+z3+z5+2\epsilon+a_1+a_2+a_3+a_4]\,(m_1^2)^{z3}\,(m_2^2)^{z1}\,(m_3^2)^{z2}\,(m_4^2)^{z5}\big)\Big/$$
$$(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[4-z1-z2-2\epsilon-a_2-a_3]\,\mathrm{Gamma}[a_3]\,\mathrm{Gamma}[6-z1-z2-z3-z5-3\epsilon-a_1-a_2-a_3-a_4]\,\mathrm{Gamma}[a_4]\,\mathrm{Gamma}[-2+z1+z2+z5+\epsilon+a_2+a_3+a_4]))$$

Obtained an MB representation of 4 fold.

The scales of the MB representation are $\left\{\frac{m_2^2}{s}, \frac{m_3^2}{s}, \frac{m_1^2}{s}, \frac{m_4^2}{s}\right\}$.

The associated $\mathcal{A}$-matrix → $\begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, which has codim = 4.

Time Taken 0.11723 seconds

Two-loop self-energy with four propagators (contd.)

Compute the unimodular regular triangulations (results shown till the 4^{th} triangulation; there are 24 in total):

Finding Triangulations

```
In[◦]:= Triangulations = FindTriangulations[FindAMatrixOut];

Finding all regular triangulations ...
Found 24 Regular Triangulations, out of which 24 are Unimodular.
The 24 Unimodular Regular Triangulations →
1 :: {{1, 2, 3, 4, 5, 6, 8, 9, 10, 11}, {1, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {1, 2, 3, 5, 6, 8, 9, 10, 11, 12}, {1, 2, 3, 4, 5, 6, 9, 10, 11, 12}, {1, 3, 5, 6, 7, 8, 9, 10, 11, 12},
  {1, 3, 4, 5, 6, 7, 9, 10, 11, 12}, {1, 2, 3, 4, 5, 6, 8, 9, 10, 13}, {1, 3, 4, 5, 6, 7, 8, 9, 10, 13}, {1, 2, 3, 5, 6, 8, 9, 10, 12, 13}, {1, 2, 3, 4, 5, 6, 9, 10, 12, 13},
  {1, 3, 5, 6, 7, 8, 9, 10, 12, 13}, {1, 3, 4, 5, 6, 7, 9, 10, 12, 13}, {1, 2, 3, 4, 5, 6, 8, 10, 11, 14}, {1, 3, 4, 5, 6, 7, 8, 10, 11, 14}, {2, 3, 5, 6, 8, 9, 10, 11, 12, 14},
  {1, 2, 3, 5, 8, 9, 10, 11, 12, 14}, {1, 2, 3, 5, 6, 8, 10, 11, 12, 14}, {1, 2, 3, 4, 5, 6, 10, 11, 12, 14}, {3, 5, 6, 7, 8, 9, 10, 11, 12, 14},
  {1, 3, 5, 7, 8, 9, 10, 11, 12, 14}, {1, 3, 5, 6, 7, 8, 10, 11, 12, 14}, {1, 3, 4, 5, 6, 7, 10, 11, 12, 14}, {1, 2, 3, 4, 5, 6, 8, 10, 13, 14}, {1, 3, 4, 5, 6, 7, 8, 10, 13, 14},
  {2, 3, 5, 6, 8, 9, 10, 12, 13, 14}, {1, 2, 3, 5, 8, 9, 10, 12, 13, 14}, {1, 2, 3, 5, 6, 8, 10, 12, 13, 14}, {1, 2, 3, 4, 5, 6, 10, 12, 13, 14},
  {3, 5, 6, 7, 8, 9, 10, 12, 13, 14}, {1, 3, 5, 7, 8, 9, 10, 12, 13, 14}, {1, 3, 5, 6, 7, 8, 10, 12, 13, 14}, {1, 3, 4, 5, 6, 7, 10, 12, 13, 14}}
2 :: {{2, 3, 5, 6, 8, 9, 10, 11, 12, 14}, {1, 2, 3, 5, 8, 9, 10, 11, 12, 14}, {3, 5, 6, 7, 8, 9, 10, 11, 12, 14}, {1, 3, 5, 7, 8, 9, 10, 11, 12, 14},
  {2, 3, 5, 6, 8, 9, 10, 12, 13, 14}, {1, 2, 3, 5, 8, 9, 10, 12, 13, 14}, {3, 5, 6, 7, 8, 9, 10, 12, 13, 14}, {1, 3, 5, 7, 8, 9, 10, 12, 13, 14},
  {2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {2, 3, 5, 6, 7, 8, 9, 10, 11, 12}, {2, 3, 4, 5, 6, 7, 9, 10, 11, 12}, {2, 3, 4, 5, 6, 7, 8, 9, 10, 13}, {2, 3, 5, 6, 7, 8, 9, 10, 12, 13},
  {2, 3, 4, 5, 6, 7, 9, 10, 12, 13}, {2, 3, 4, 5, 6, 7, 8, 10, 11, 14}, {2, 3, 5, 6, 7, 8, 10, 11, 12, 14}, {2, 3, 4, 5, 6, 7, 10, 11, 12, 14}, {2, 3, 4, 5, 6, 7, 8, 10, 13, 14},
  {2, 3, 5, 6, 7, 8, 10, 12, 13, 14}, {2, 3, 4, 5, 6, 7, 10, 12, 13, 14}, {1, 2, 3, 4, 5, 7, 8, 9, 10, 11}, {1, 2, 3, 5, 7, 8, 9, 10, 11, 12}, {1, 2, 3, 4, 5, 7, 9, 10, 11, 12},
  {1, 2, 3, 4, 5, 7, 8, 9, 10, 13}, {1, 2, 3, 5, 7, 8, 9, 10, 12, 13}, {1, 2, 3, 4, 5, 7, 9, 10, 12, 13}, {1, 2, 3, 4, 5, 7, 8, 10, 11, 14}, {1, 2, 3, 5, 7, 8, 10, 11, 12, 14},
  {1, 2, 3, 4, 5, 7, 10, 11, 12, 14}, {1, 2, 3, 4, 5, 7, 8, 10, 13, 14}, {1, 2, 3, 5, 7, 8, 10, 12, 13, 14}, {1, 2, 3, 4, 5, 7, 10, 12, 13, 14}}
3 :: {{1, 2, 3, 4, 5, 6, 8, 9, 10, 11}, {1, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {1, 2, 3, 4, 5, 6, 9, 10, 11, 12}, {1, 3, 4, 5, 6, 7, 9, 10, 11, 12}, {1, 2, 3, 4, 5, 6, 8, 9, 10, 13},
  {1, 3, 4, 5, 6, 7, 8, 9, 10, 13}, {1, 2, 3, 4, 5, 6, 9, 10, 12, 13}, {1, 3, 4, 5, 6, 7, 9, 10, 12, 13}, {1, 2, 3, 4, 5, 6, 8, 10, 11, 14}, {1, 3, 4, 5, 6, 7, 8, 10, 11, 14},
  {1, 2, 3, 4, 5, 6, 10, 11, 12, 14}, {1, 3, 4, 5, 6, 7, 10, 11, 12, 14}, {1, 2, 3, 4, 5, 6, 8, 10, 13, 14}, {1, 3, 4, 5, 6, 7, 8, 10, 13, 14}, {1, 2, 3, 4, 5, 6, 10, 12, 13, 14},
  {1, 3, 4, 5, 6, 7, 10, 12, 13, 14}, {1, 2, 5, 6, 8, 9, 10, 11, 12, 14}, {1, 5, 6, 7, 8, 9, 10, 11, 12, 14}, {1, 2, 5, 6, 8, 9, 10, 12, 13, 14},
  {1, 5, 6, 7, 8, 9, 10, 12, 13, 14}, {1, 2, 3, 6, 8, 9, 10, 11, 12, 14}, {1, 3, 6, 7, 8, 9, 10, 11, 12, 14}, {1, 2, 3, 6, 8, 9, 10, 12, 13, 14},
  {1, 3, 6, 7, 8, 9, 10, 12, 13, 14}, {1, 2, 3, 5, 6, 9, 10, 11, 12, 14}, {1, 3, 5, 6, 7, 9, 10, 11, 12, 14}, {1, 2, 3, 5, 6, 9, 10, 12, 13, 14},
  {1, 3, 5, 6, 7, 9, 10, 12, 13, 14}, {1, 2, 3, 5, 6, 8, 9, 10, 11, 14}, {1, 3, 5, 6, 7, 8, 9, 10, 11, 14}, {1, 2, 3, 5, 6, 8, 9, 10, 13, 14}, {1, 3, 5, 6, 7, 8, 9, 10, 13, 14}}
4 :: {{1, 2, 4, 5, 6, 8, 9, 10, 11, 13}, {1, 4, 5, 6, 7, 8, 9, 10, 11, 13}, {1, 2, 5, 6, 8, 9, 10, 11, 12, 13}, {1, 2, 4, 5, 6, 9, 10, 11, 12, 13}, {1, 5, 6, 7, 8, 9, 10, 11, 12, 13},
  {1, 4, 5, 6, 7, 9, 10, 11, 12, 13}, {1, 2, 4, 5, 6, 8, 10, 11, 13, 14}, {1, 4, 5, 6, 7, 8, 10, 11, 13, 14}, {2, 5, 6, 8, 9, 10, 11, 12, 13, 14}, {1, 2, 5, 8, 9, 10, 11, 12, 13, 14},
  {1, 2, 5, 6, 8, 10, 11, 12, 13, 14}, {1, 2, 4, 5, 6, 10, 11, 12, 13, 14}, {5, 6, 7, 8, 9, 10, 11, 12, 13, 14}, {1, 5, 7, 8, 9, 10, 11, 12, 13, 14},
  {1, 5, 6, 7, 8, 10, 11, 12, 13, 14}, {1, 4, 5, 6, 7, 10, 11, 12, 13, 14}, {1, 2, 3, 4, 6, 8, 9, 10, 11, 13}, {1, 3, 4, 6, 7, 8, 9, 10, 11, 13}, {1, 2, 3, 6, 8, 9, 10, 11, 12, 13},
  {1, 2, 3, 4, 6, 9, 10, 11, 12, 13}, {1, 3, 6, 7, 8, 9, 10, 11, 12, 13}, {1, 3, 4, 6, 7, 9, 10, 11, 12, 13}, {1, 2, 3, 4, 6, 8, 10, 11, 13, 14}, {1, 3, 4, 6, 7, 8, 10, 11, 13, 14},
  {2, 3, 6, 8, 9, 10, 11, 12, 13, 14}, {1, 2, 3, 8, 9, 10, 11, 12, 13, 14}, {1, 2, 3, 6, 8, 10, 11, 12, 13, 14}, {1, 2, 3, 4, 6, 10, 11, 12, 13, 14},
  {3, 6, 7, 8, 9, 10, 11, 12, 13, 14}, {1, 3, 7, 8, 9, 10, 11, 12, 13, 14}, {1, 3, 6, 7, 8, 10, 11, 12, 13, 14}, {1, 3, 4, 6, 7, 10, 11, 12, 13, 14}}
```

Two-loop self-energy with four propagators (contd.)

Calculate the Γ-series (three of the terms contributing to the full solution for the 4^{th} unimodular regular triangulation have been shown here):

Finding Analytic Series

```
In[∘]:= SeriesSolution = SeriesRepresentation[Triangulations, 4];
```

Unimodular Triangulation → 4

Number of summation variables → 4

The series solution is the sum of following 16 terms.

Term 1 ::

$$\Big(\Big(s^{4-2\epsilon-a_1-a_2-a_3-a_4}\,\mathrm{Gamma}[2-\epsilon-a_2-n_1]\,\mathrm{Gamma}[2-\epsilon-a_1-n_2]\,\mathrm{Gamma}[a_4+n_3]\,\mathrm{Gamma}[-2+\epsilon+a_3-n_4]\,\mathrm{Gamma}[2-\epsilon-a_2-a_4-n_1-n_3-n_4]\,\mathrm{Gamma}[a_2+n_1+n_4]\,\mathrm{Gamma}[-2+\epsilon+a_1+a_2+a_4+n_1+n_2+n_3+n_4]\left(-\frac{m_1^2}{s}\right)^{n_2}\left(-\frac{m_2^2}{s}\right)^{n_1}\left(-\frac{m_3^2}{s}\right)^{n_4}\left(\frac{m_3^2}{s}\right)^{2-\epsilon-a_3}\left(-\frac{m_4^2}{s}\right)^{n_3}\Big)\Big/\big(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[a_3]\,\mathrm{Gamma}[a_4]\,\mathrm{Gamma}[1+n_1]\,\mathrm{Gamma}[1+n_2]\,\mathrm{Gamma}[1+n_3]\,\mathrm{Gamma}[2-\epsilon-a_2-n_1-n_4]\,\mathrm{Gamma}[4-2\epsilon-a_1-a_2-a_4-n_1-n_2-n_3-n_4]\,\mathrm{Gamma}[1+n_4]\,\mathrm{Gamma}[a_2+a_4+n_1+n_3+n_4]\big)\Big)$$

Term 2 ::

$$\Big(\Big(s^{4-2\epsilon-a_1-a_2-a_3-a_4}\,\mathrm{Gamma}[2-\epsilon-a_1-n_1]\,\mathrm{Gamma}[a_4+n_2]\,\mathrm{Gamma}[-2+\epsilon+a_2-n_3]\,\mathrm{Gamma}[-2+\epsilon+a_3-n_4]\,\mathrm{Gamma}[-a_4-n_2-n_3-n_4]\,\mathrm{Gamma}[2-\epsilon+n_3+n_4]\,\mathrm{Gamma}[a_1+a_4+n_1+n_2+n_3+n_4]\left(-\frac{m_1^2}{s}\right)^{n_1}\left(-\frac{m_2^2}{s}\right)^{n_3}\left(\frac{m_2^2}{s}\right)^{2-\epsilon-a_2}\left(-\frac{m_3^2}{s}\right)^{n_4}\left(\frac{m_3^2}{s}\right)^{2-\epsilon-a_3}\left(-\frac{m_4^2}{s}\right)^{n_2}\Big)\Big/\big(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[a_3]\,\mathrm{Gamma}[a_4]\,\mathrm{Gamma}[1+n_1]\,\mathrm{Gamma}[1+n_2]\,\mathrm{Gamma}[1+n_3]\,\mathrm{Gamma}[-n_3-n_4]\,\mathrm{Gamma}[2-\epsilon-a_1-a_4-n_1-n_2-n_3-n_4]\,\mathrm{Gamma}[1+n_4]\,\mathrm{Gamma}[2-\epsilon+a_4+n_2+n_3+n_4]\big)\Big)$$

Term 3 ::

$$\Big(\Big(s^{4-2\epsilon-a_1-a_2-a_3-a_4}\,\mathrm{Gamma}[2-\epsilon-a_1-n_1]\,\mathrm{Gamma}[a_4+n_2]\,\mathrm{Gamma}[-2+\epsilon+a_3-n_3]\,\mathrm{Gamma}[a_4+n_2+n_3-n_4]\,\mathrm{Gamma}[-2+\epsilon+a_2+a_4+n_2+n_3-n_4]\,\mathrm{Gamma}[a_1+n_1+n_4]\,\mathrm{Gamma}[2-\epsilon-a_4-n_2+n_4]\left(-\frac{m_1^2}{s}\right)^{n_1}\left(-\frac{m_2^2}{s}\right)^{n_4}\left(\frac{m_2^2}{s}\right)^{2-\epsilon-a_2-a_4}\left(\frac{m_3^2}{s}\right)^{2-\epsilon-a_3}\left(-\frac{m_3^2}{m_2^2}\right)^{n_3}\left(-\frac{m_4^2}{m_2^2}\right)^{n_2}\Big)\Big/\big(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[a_3]\,\mathrm{Gamma}[a_4]\,\mathrm{Gamma}[1+n_1]\,\mathrm{Gamma}[1+n_2]\,\mathrm{Gamma}[1+n_3]\,\mathrm{Gamma}[2-\epsilon-a_1-n_1-n_4]\,\mathrm{Gamma}[a_4+n_2-n_4]\,\mathrm{Gamma}[1+n_4]\,\mathrm{Gamma}[2-\epsilon+n_4]\big)\Big)$$

Two-loop self-energy with four propagators (contd.)

Evaluate the sum of the Γ-series terms numerically:

Numerical Analysis

```
In[∘]:= SumLim = 15;
ParameterSub = {ϵ → 0.001, a₁ → 1, a₂ → 1, a₃ → 7 / 8, a₄ → 3 / 4, m₁ → 0.1, m₂ → 5, m₃ → 0.3, m₄ → 0.3, s → 100};
NumericalSum[SeriesSolution, ParameterSub, SumLim, RunInParallel → True];

Numerical result = 64.7166
Time Taken 336.024 seconds

In[∘]:= FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];

FIESTA Value = 64.7165
Time Taken 179.623 seconds
```

Summary and Future Works

Summary

- Very rich mathematical structures appear in context of computing multiloop multiscale Feynman integrals.
- In-depth analysis of such structures might furnish insights for developing novel computational frameworks and algorithms for evaluating these integrals.
- Laurent expansion of hypergeometric functions in the dim-reg parameter ϵ - critical bottleneck in this approach. Known automated implementations: `HypExp`, `XSummer`, also private in-house implementations.

Future Works

- Tackle the issue of ϵ-expansion of multivariate hypergeometric functions[16]
- Convert the standing proof-of-concept implementation into a performance-driven one. Current performance bottlenecks stem from an excess of dependence on *Mathematica*, particularly in the numerical summation step.
- Explore the scope of `FeynGKZ` in evaluating stringy canonical forms[17]

[16] Bera 2022, 2024; Bezuglov et al. 2023.
[17] He et al. 2020.

Future Works (contd.)

- Extending studies of the analytic structure of Feynman integrals in the GKZ formalism[18]. Why? Extracting the symbol alphabet from the integral representation by analysing the Landau singularities[19] instead of going through the traditional IBP-DE route could help tackle computational challenges[20]

[18] Klausen 2020.
[19] Dlapa et al. 2023.
[20] Abreu et al. 2020.

Acknowledgements

Acknowledgements

I thank my collaborators, and in particular Narayan da for organising the first-ever ASWMSA workshop. I also thank all the groups working around the world to produce remarkable open-source computer codes on novel mathematical structures, without which, this work would not have been possible!

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