Lecture I:
Basics of subtraction methods

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Advanced School & Workshop on Multiloop Scattering Amplitudes

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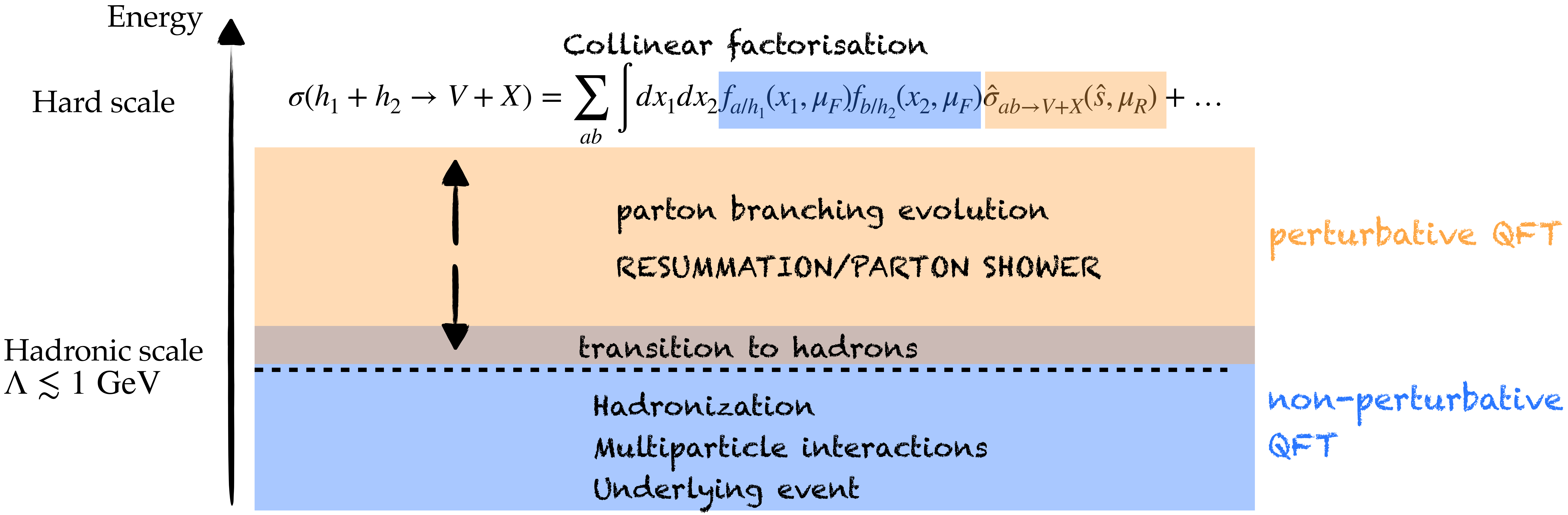
Radiative corrections for LHC phenomenology

Hadron-hadron collisions: very complicated processes probing multi-scale nature of QFT in perturbative and non-perturbative regimes

Collinear factorisation

$$\sigma(h_1 + h_2 \to V + X) = \sum_{ab} \int dx_1 dx_2 f_{a/h_1}(x_1, \mu_F) f_{b/h_2}(x_2, \mu_F) \hat{\sigma}_{ab \to V+X}(\hat{s}, \mu_R) + \dots$$



Radiative corrections for LHC phenomenology

Hadron-hadron collisions: very complicated processes probing multi-scale nature of QFT in perturbative and non-perturbative regimes

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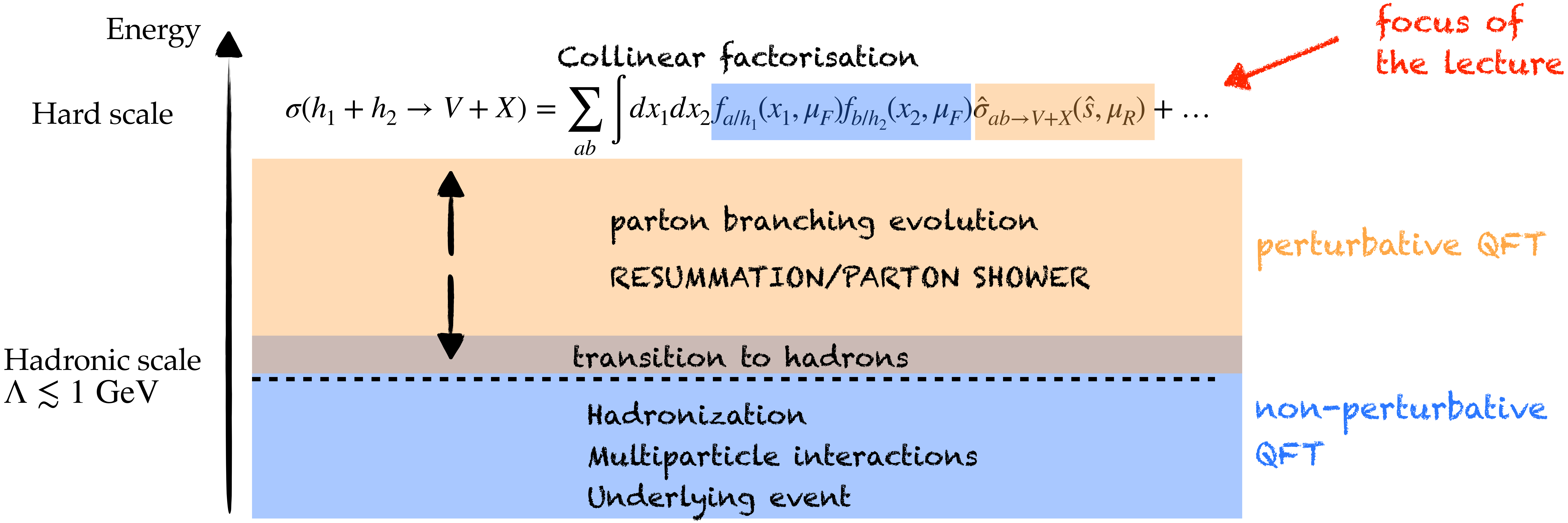
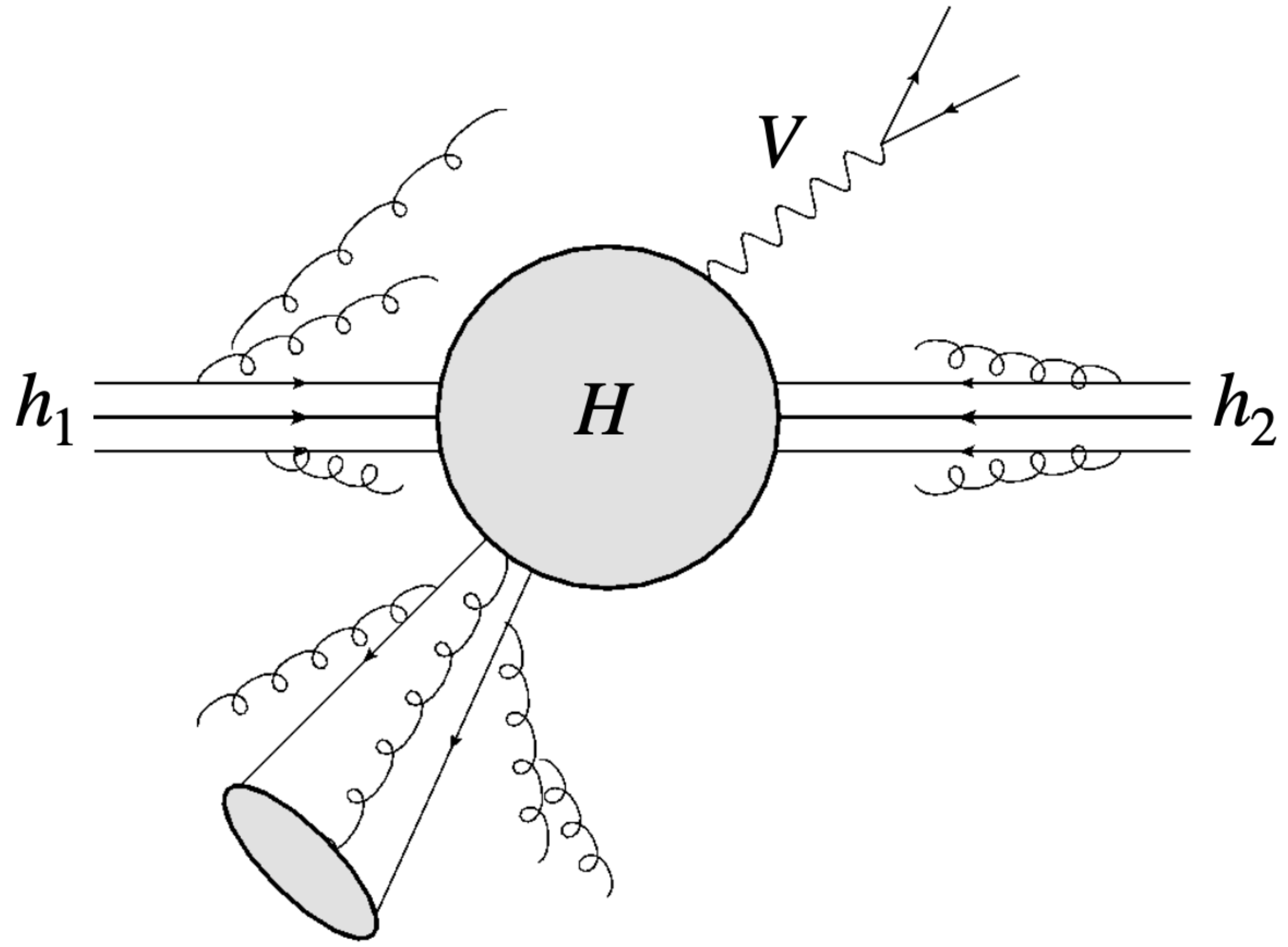


Hard Scattering: fixed Order Predictions



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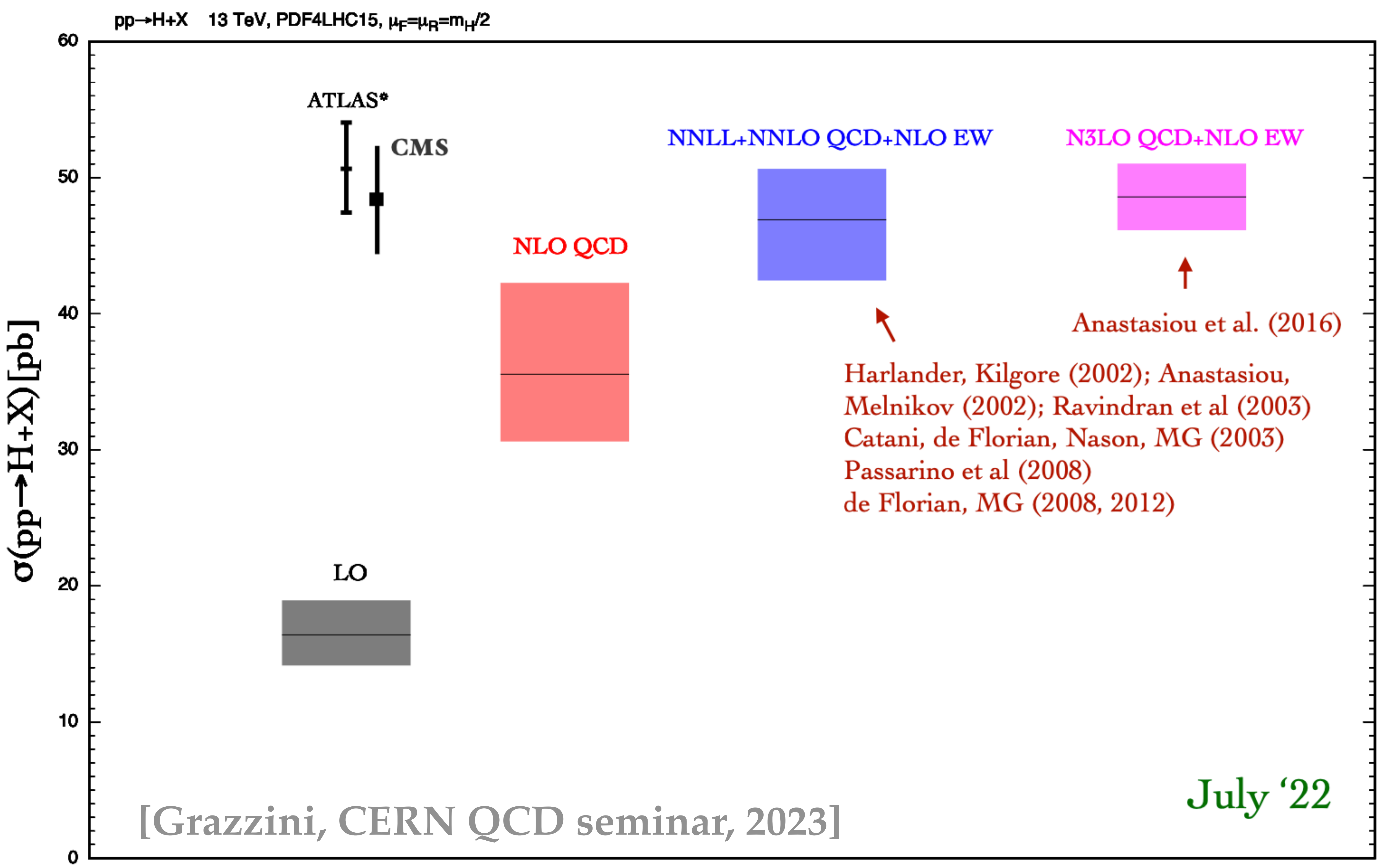
Elementary partonic cross section can be computed in perturbation theory

$$\hat{\sigma}_{ab} = \hat{\sigma}^{(0)}_{ab} + \frac{\alpha_S}{2\pi} \hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_S}{2\pi} \right)^2 \hat{\sigma}^{(2)}_{ab} + \dots$$

$\mathcal{O}(100\%)$ $\mathcal{O}(20\%)$ $\mathcal{O}(5\%)$

LO NLO NNLO

Hard Scattering: higher orders at work!



using pseudo data with nominal top mass $m_t = 174.3$ GeV

TH. ACC.	m_t [χ^2]
NLO+PS+MS	$174.48^{+0.73}_{-0.77}$ [5.0]
LO+PS+MS	$175.98^{+0.63}_{-0.69}$ [16.9]
NLO+PS	$175.43^{+0.74}_{-0.80}$ [29.2]
LO+PS	$187.90^{+0.6}_{-0.6}$ [428.3]
fNLO	$174.41^{+0.72}_{-0.73}$ [96.6]
fLO	$197.31^{+0.42}_{-0.35}$ [2496.1]

[Frixione, Mitov, 2014]

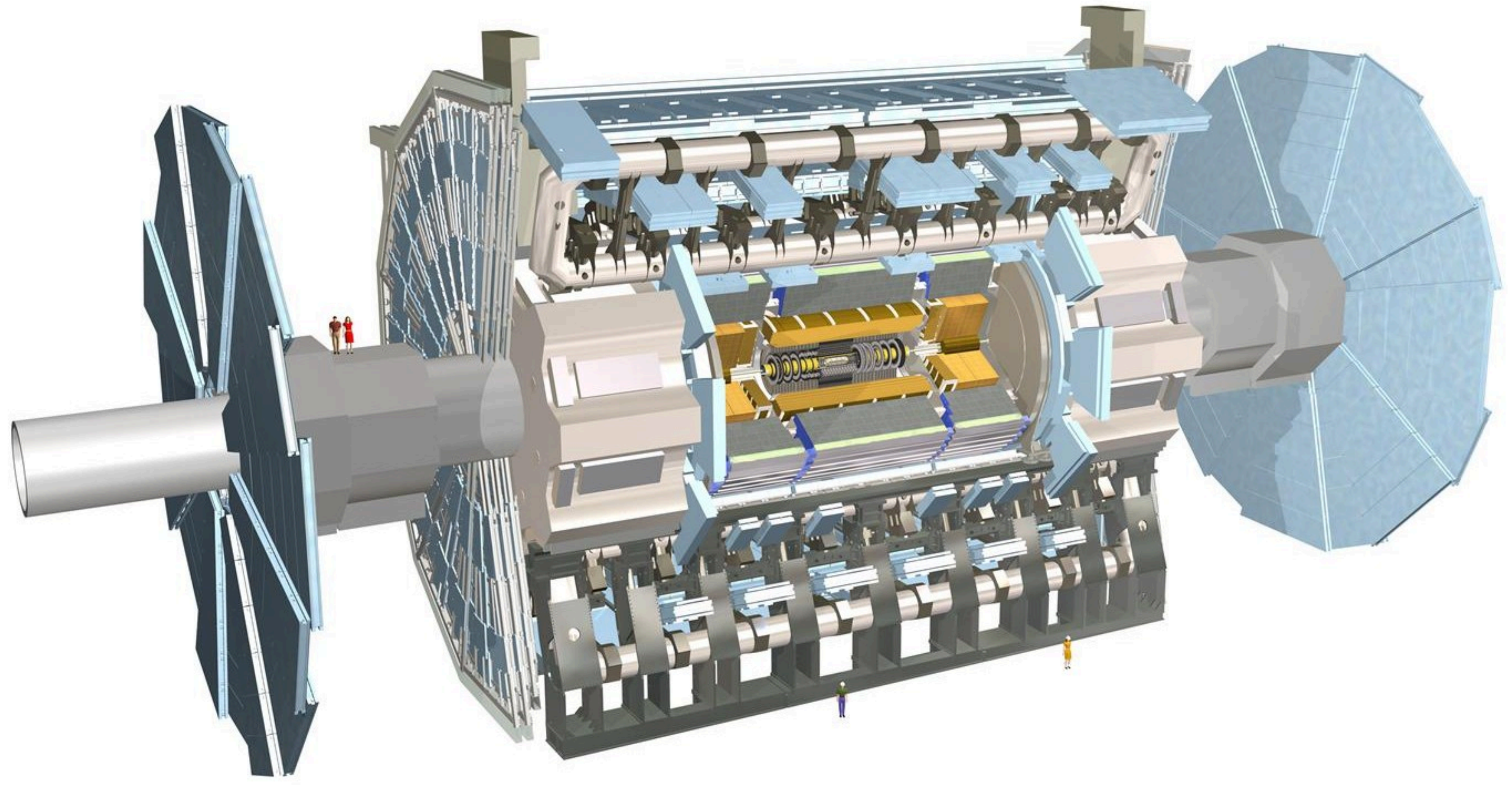
Higgs boson discovery: emblematic case of the importance of higher-order corrections

Basically, LO ruled out by experiment

Extracting theory parameters from measurements can depend on the "theory model" employed, including the perturbative order used!

Hard Scattering: LO & Monte Carlo integration

$$\hat{\sigma}_{ab}^{(0)} = \int d\Phi_n \, |M_B(\Phi_n)|^2$$



Integration becomes **soon intractable** with analytical methods

- high-dimensional integration scaling as $3n - 4$
- experimental requirements (fiducial volume), differential distributions, jet clustering, isolation…

MONTE CARLO integration as weighted average over a sample of **events** $\{\Phi_n^i\}_{i=1}^N$ in phase space

$$< \mathcal{O} > = \int d\Phi_n \, |M_B(\Phi_n)|^2 F_{\mathcal{O}}^n(\Phi_n) \simeq \frac{1}{N} \sum_i J(\Phi_n^i) \, |M_B(\Phi_n^i)|^2 F_{\mathcal{O}}^n(\Phi_n^i)$$

$$\Phi_n^i = (p_1^i, \ldots, p_n^i)$$
$$w^i = J(\Phi_n^i) \, |M_B(\Phi_n^i)|^2 F_{\mathcal{O}}^n(\Phi_n^i)$$

→

if the **event** lie in the j-th bin of a multi-dimensional histogram $\{h_l\}$ then increase $h_j = h_j + w^i$

Hard Scattering: @ NLO

At LO numerical approach straightforward as there are **no exceptional configurations** (may require a suitable definition of the cross section)

$$< \mathcal{O} > = \int d\Phi_n \left[(|M_B(\Phi_n)|^2 + 2\Re(M_V M_B^*)(\Phi_n) \right] F_{\mathcal{O}}^n(\Phi_n) + \int d\Phi_{n+1} |M_R(\Phi_{n+1})|^2 F_{\mathcal{O}}^{n+1}(\Phi_{n+1})$$

UV renormalised virtual amplitude: divergent in infrared and/or collinear (IRC) limits exposed as explicit poles in dimensional regularisation

Real emission amplitude: divergent upon integration over phase space when two massless partons become collinear and/or one parton become soft

BN and KNL theorems ensure cancellation of divergences for IRC-safe observables $\mathcal{O}$, but requires an **analytical treatment of the integration which becomes soon intractable**

Hard Scattering: NLO

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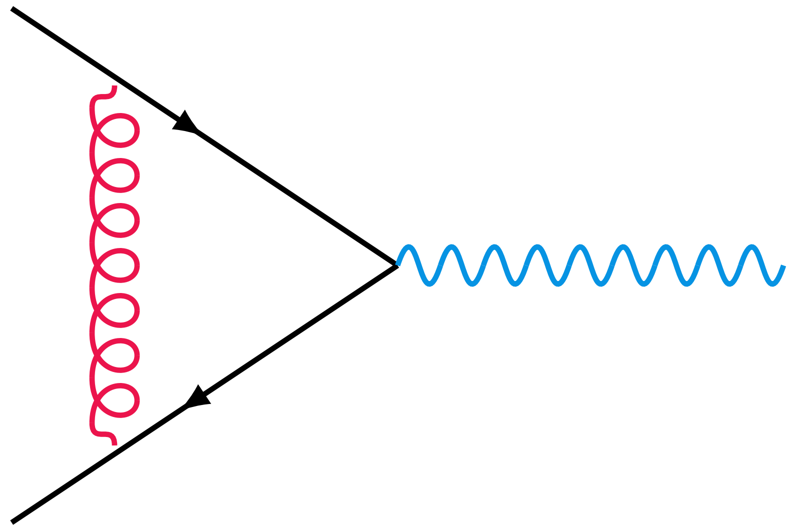

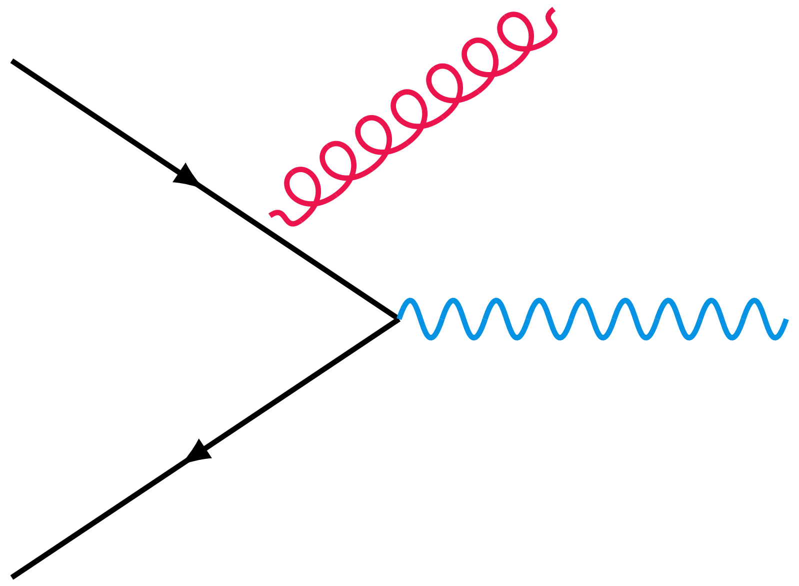

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UV renormalised virtual amplitude:
divergent in infrared and/or collinear (IRC) limits exposed as explicit poles in dimensional regularisation

Real emission amplitude:
divergent upon integration over phase space when two massless partons become collinear and/or one parton become soft

ISSUE: Monte Carlo integration required; how to achieve the cancellation of intermediate singularities while retaining the flexibility of the numerical approach?

Outline

ISSUE: Monte Carlo integration required; how to achieve the cancellation of intermediate singularities while retaining the flexibility of the numerical approach?

@ NLO

- toy-model example
- FKS approach
- CS approach

@NNLO

- anatomy of the complications

Remarks

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Toy model @ NLO: inclusive calculation

Consider a toy model of a NLO calculation with only one singular (soft) region

- the Real phase space is given by the one-dimensional interval [0,1] and the Real matrix element develops a logarithmic singularity as $x \to 0$ (soft limit) regulated in dimensional regularisation
- the Born (and Virtual) phase space is fully constrained (for example by momentum conservation)

$$\sigma_V = \frac{A}{\epsilon} + B$$

$$\sigma_R = \int_0^1 dx \frac{A + Cx}{x^{1+\epsilon}} = \left[-A\frac{x^{-\epsilon}}{\epsilon} + C\frac{x^{1-\epsilon}}{1-\epsilon} \right]_0^1 \quad \text{assume } \epsilon < 0$$

$$= -\frac{A}{\epsilon} + C + \mathcal{O}(\epsilon)$$

$$\sigma = \lim_{\epsilon \to 0} \left[\sigma_V + \sigma_R \right] = \frac{\not{A}}{\not{\epsilon}} + B - \frac{\not{A}}{\not{\epsilon}} + C = A + C \quad \text{finite!}$$

Comments

- **Virtual contribution:** integration over the loop momentum leads to **explicit** poles in ϵ
- **Real contribution:** poles in ϵ arising from phase space integration
- **Analytic cancellation** of poles

Toy model @ NLO: let's go differential!

Consider a toy model of a NLO calculation with only one singular (soft) region

- $\hat{\mathcal{O}}$ is an infrared and collinear (IRC) observable, for example a bin of a well defined kinematical histogram with/or a collection of requirements (acceptance, jet algorithm, isolation)
- the expectation value for $\hat{\mathcal{O}}$ is obtained considering the differential cross section as probability distribution

$$< \hat{\mathcal{O}} > = \left(\frac{A}{\epsilon} + B\right) F_{\hat{\mathcal{O}}}(0) + \int_0^1 dx \frac{A + Cx}{x^{1+\epsilon}} F_{\hat{\mathcal{O}}}(x)$$

$F_{\hat{\mathcal{O}}}(x)$ is the measurement function associated to $\hat{\mathcal{O}}$

$$\lim_{x \to 0} F_{\hat{\mathcal{O}}}(x) = F_{\hat{\mathcal{O}}}(0)$$

IRC condition for $F_{\hat{\mathcal{O}}}(x)$

The integral can be hard (impossible?) to do analytically for a generic measurement function

Numerical (Monte Carlo) integration would be a more **flexible** solution.

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ISSUE: (efficiently) handle the singularity in ϵ in a numerical scheme

IDEA: *split* the real integration into a **complex but integrable** piece (to be performed *numerically*) and a divergent but simple one (to be performed *analytically*) in order to achieve the analytical cancellation of the ϵ poles

Toy model @ NLO: subtraction

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SUBTRACTION: the art of adding zeros

Counterterm

- encodes the **divergent behaviour**
- integral **independent from** $F_{\hat{\mathcal{O}}}$
- simple enough for **analytical integration**

$$\int_0^1 dx \frac{A+Cx}{x^{1+\epsilon}} F_{\hat{\mathcal{O}}}(x) = \int_0^1 dx \frac{A+Cx}{x^{1+\epsilon}} \left[F_{\hat{\mathcal{O}}}(x) \underbrace{- F_{\hat{\mathcal{O}}}(0) + F_{\hat{\mathcal{O}}}(0)}_{0}\right]$$

$$= \int_0^1 dx \frac{A+Cx}{x^{1+\cancel{\epsilon}}} \left[F_{\hat{\mathcal{O}}}(x) - F_{\hat{\mathcal{O}}}(0)\right] + F_{\hat{\mathcal{O}}}(0) \int_0^1 dx \frac{A+Cx}{x^{1+\epsilon}}$$

$$= \int_0^1 dx \frac{A+Cx}{x} \left[F_{\hat{\mathcal{O}}}(x) - F_{\hat{\mathcal{O}}}(0)\right] + \left(-\frac{A}{\epsilon} + C\right) F_{\hat{\mathcal{O}}}(0) + \mathcal{O}(\epsilon)$$

Integrable, can be performed numerically

Integrated Counterterm: can be combined with the virtual contribution

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$$
\begin{aligned}
< \hat{\mathcal{O}} > &= \left(\cancel{\frac{A}{\epsilon}} + B\right) F_{\hat{\mathcal{O}}}(0) + \int_0^1 dx \frac{A + Cx}{x} \left[F_{\hat{\mathcal{O}}}(x) - F_{\hat{\mathcal{O}}}(0)\right] + \left(-\cancel{\frac{A}{\epsilon}} + C\right) F_{\hat{\mathcal{O}}}(0) \\
&= (B + C)\, F_{\hat{\mathcal{O}}}(0) + \int_0^1 dx \frac{A + Cx}{x} \left[F_{\hat{\mathcal{O}}}(x) - F_{\hat{\mathcal{O}}}(0)\right]
\end{aligned}
$$

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The calculation is reorganised in a such a way that

- the cancellation of (infrared and collinear) singularities between real and virtual contributions occurs **analytically**
- the complicated phase space integrals which encode the dependence upon the measurement function can be performed **numerically**

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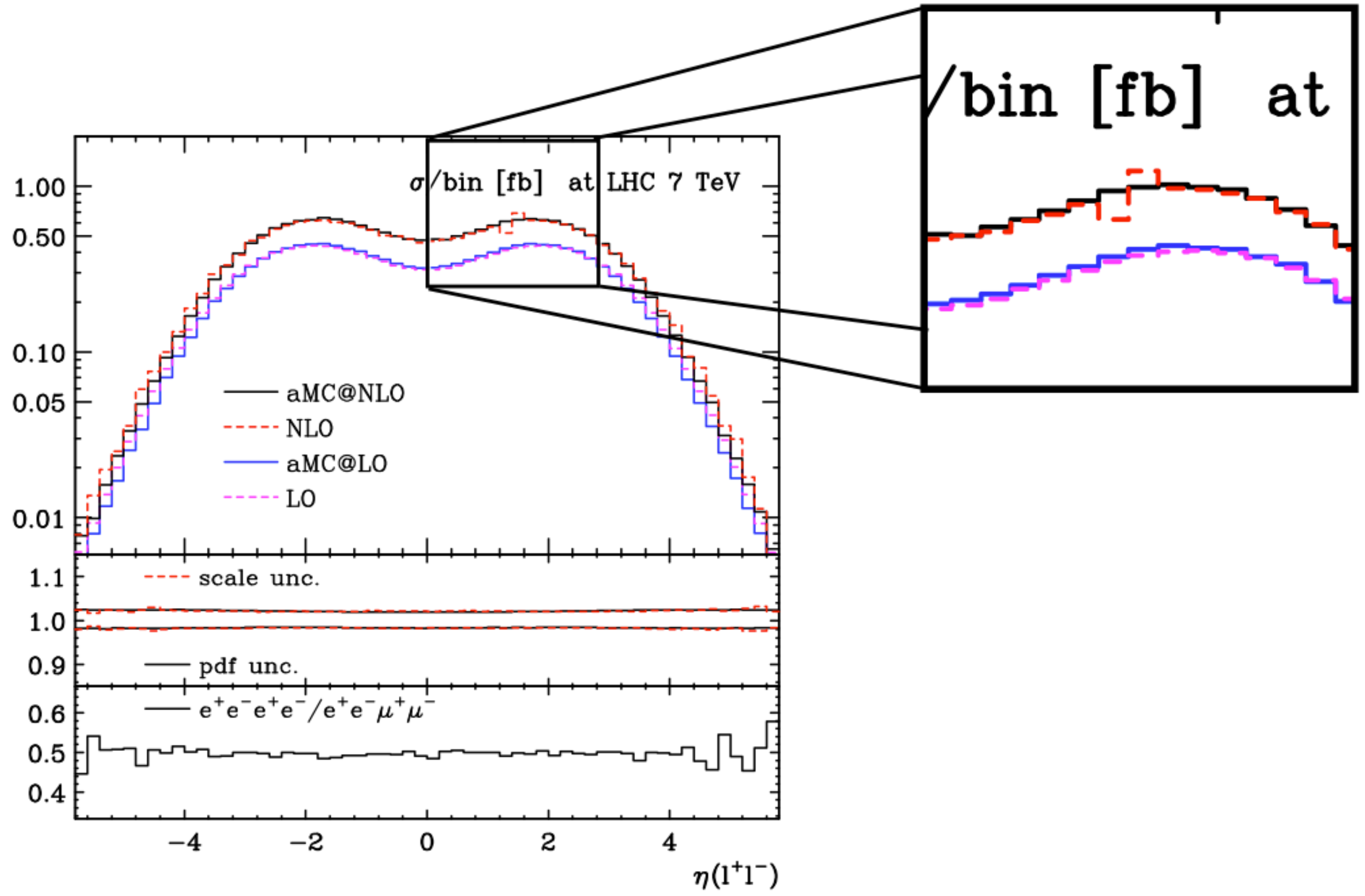

Challenges

- **loss of precision due to float arithmetic:** large cancellation between **events** and **counter-events** near the singular limit (numerical stability of amplitudes, introduction of technical cutoff)
- **mis-binning**: the weights of a pair event/counter-event may fall into two different bins. Required more statistics. At NLO it is usually under control, at higher orders it may represent a sever problem

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Subtraction @ NLO: FKS in two steps (step I)

[Frixione, Kunszt, Signer (1998)]

For a process with n parton in the final state (and, for simplicity, no identified hadrons in the initial state) at the lowest order, in general we have

$$< \hat{\mathcal{O}} > = \int d\Phi_n [B(\Phi_n) + V(\Phi_n)] F^n_{\hat{\mathcal{O}}}(\Phi_n) + \int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1})$$

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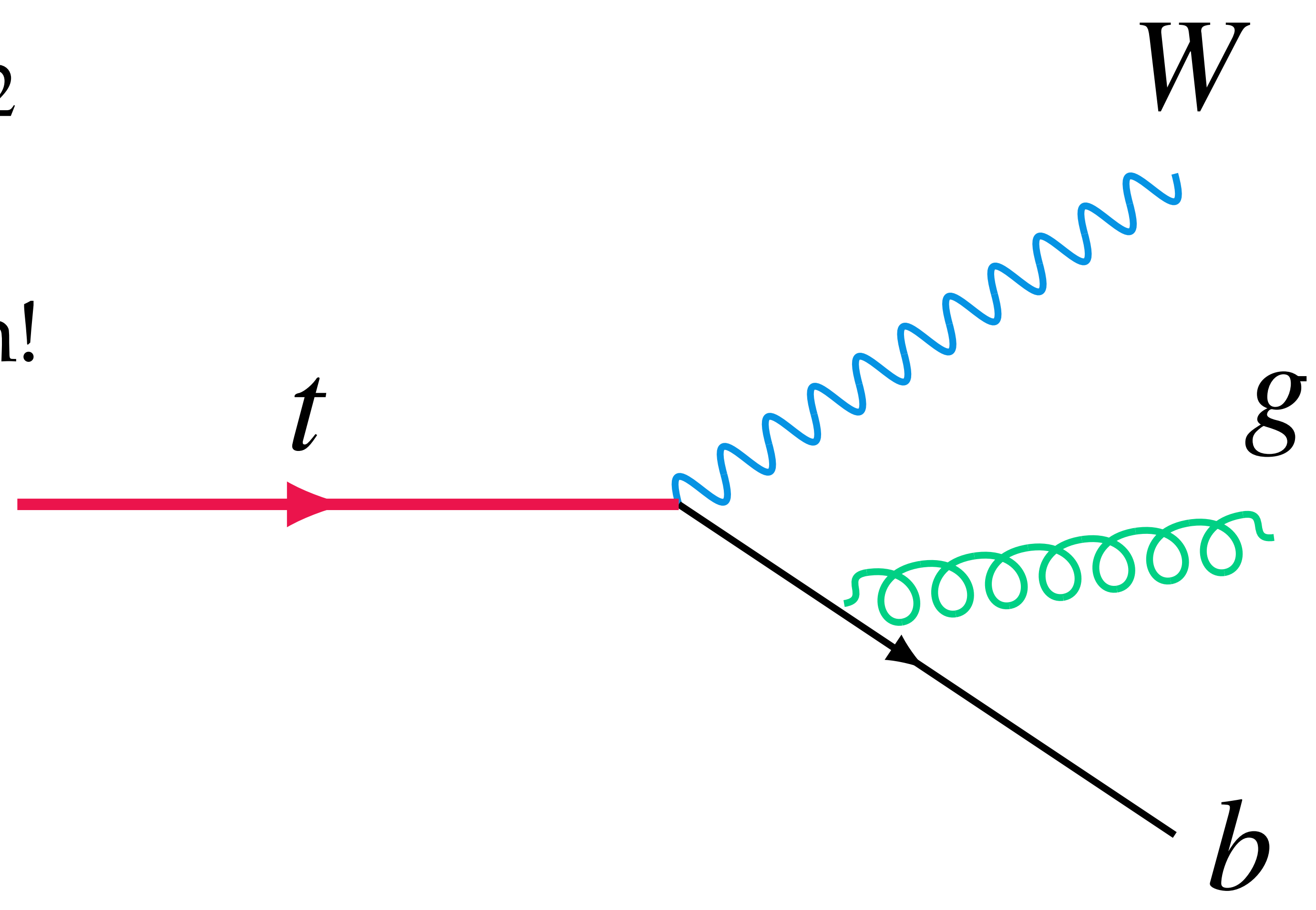

IDEA from the toy model: use the plus prescription to generate counterterm!

$$\int_0^1 dx \frac{A + Cx}{x} \left[F_{\hat{\mathcal{O}}}(x) - F_{\hat{\mathcal{O}}}(0) \right] = \int_0^1 dx \left(\frac{A + Cx}{x} \right)_+ F_{\hat{\mathcal{O}}}(x)$$

Consider a process with only one massless parton at the lowest order, for example the electroweak top decay $t \to W + b$ with a massless bottom quark

The real emission processes is $t \to W + b + g$. Then, the singular limits are

- gluon becoming parallel to the bottom quark (**collinear limit**)
- gluon becoming soft (**soft limit**)

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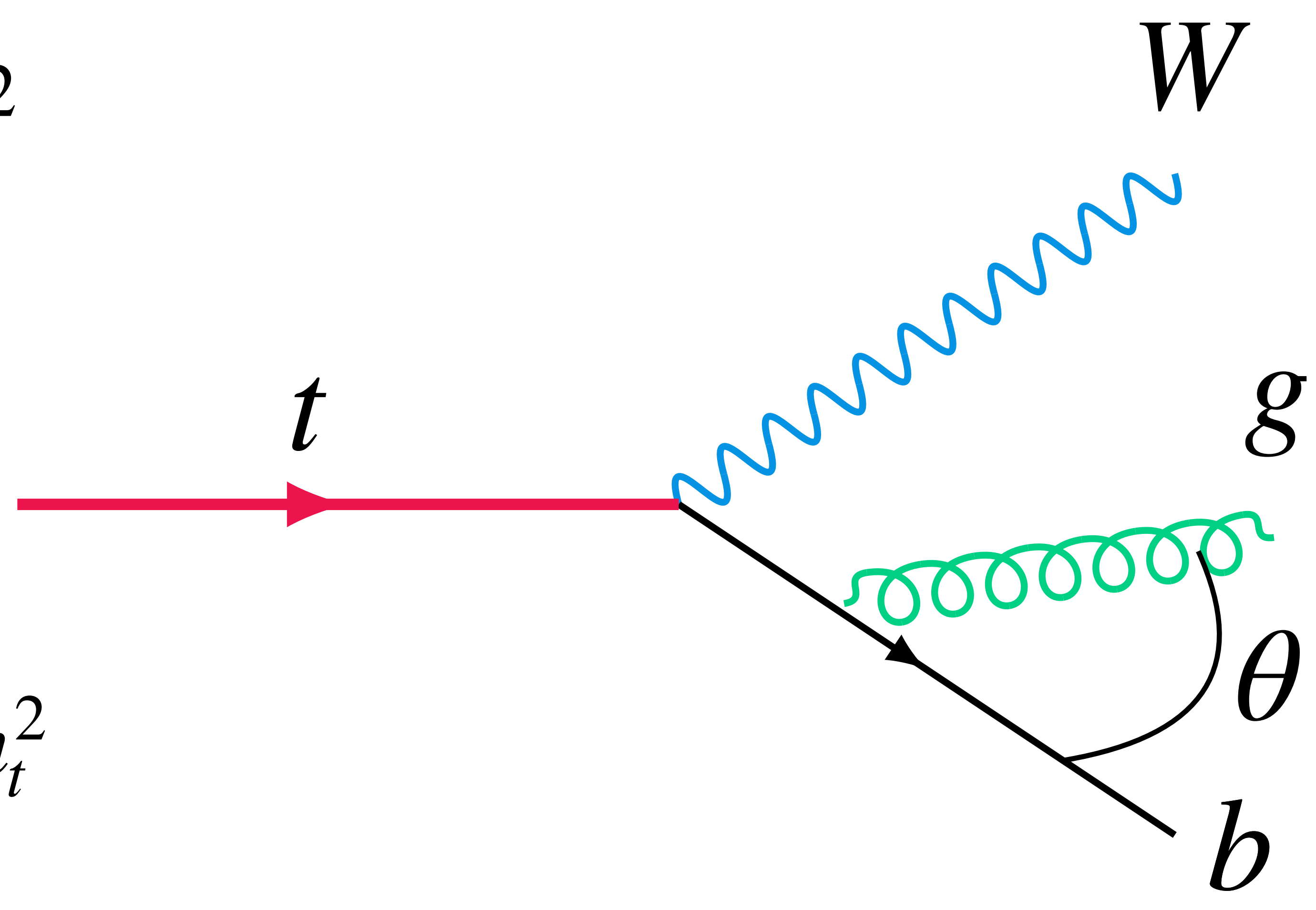

Introduce **FKS parametrisation** for the radiation phase space
(**frame dependent;** standard choice is the partonic centre of mass frame of the real configuration)

$$\Phi_{\rm rad} = \left(\xi \equiv \frac{2k_g^0}{\sqrt{s}}, y \equiv \cos\theta, \phi \right), \quad s = p_t^2 = m_t^2$$

soft limit, $\xi \to 0$ collinear limit, $y \to 1$

$$d\Phi_{\rm rad} \sim \frac{d^{d-1}k_g}{2k_g^0} = \frac{1}{2}(k^0)^{d-3} dk^0 \sin^{d-3}\theta d\theta d\Omega^{d-2} = \frac{1}{2}\left(\frac{s}{2}\right)^{1-\epsilon} \xi^{1-2\epsilon} d\xi (1-y^2)^{-\epsilon} dy d\Omega^{2-2\epsilon}$$

$$d = 4 - 2\epsilon, \quad d\Omega^{d-2} = \sin^{d-4}\phi d\phi d\Omega^{d-3}$$

[Frixione, Nason, Oleari, (2007)]

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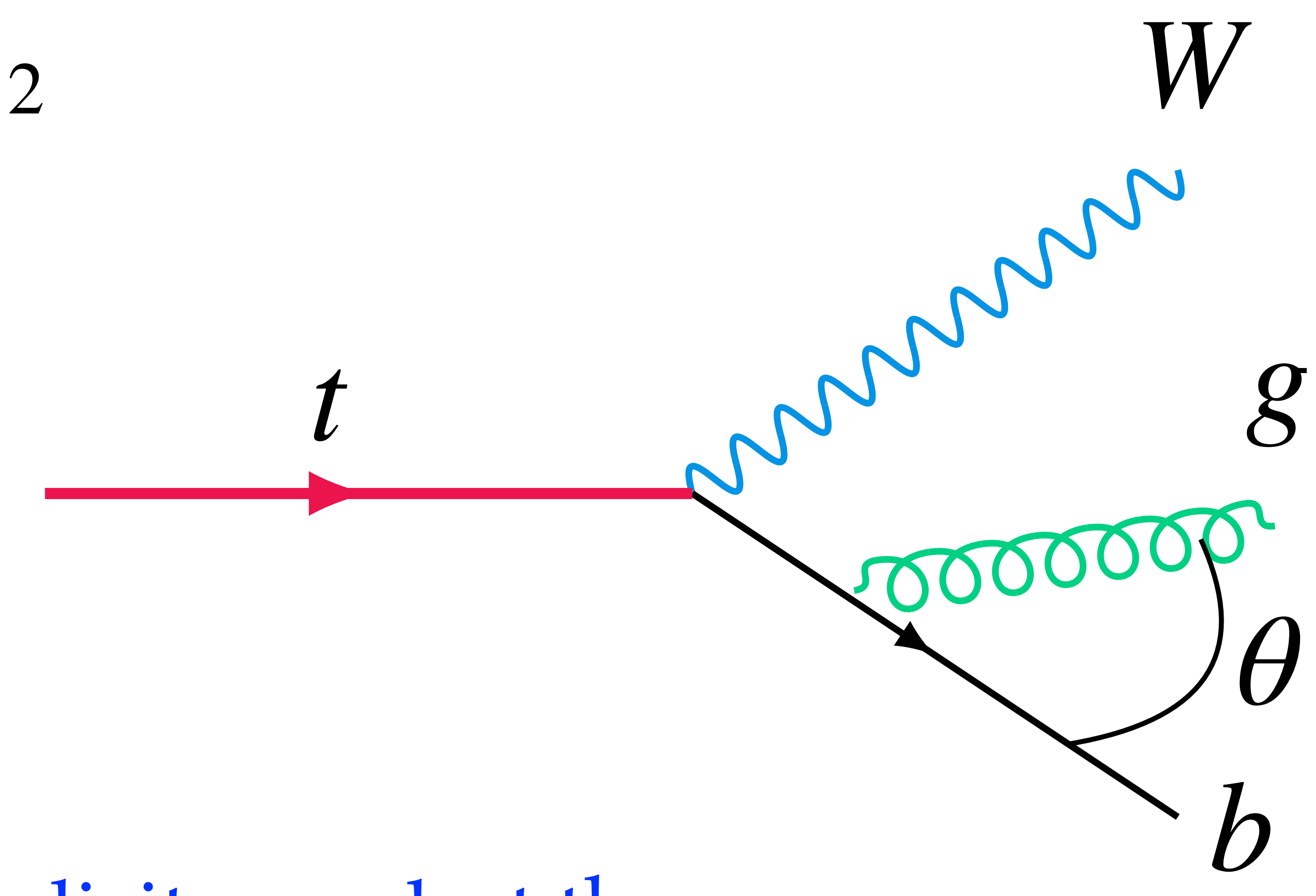

$$d\Phi_{\rm rad} \sim \xi^{1-2\epsilon} d\xi (1-y)^{-\epsilon} dy$$

phase space vanishes as ξ in the soft $\xi \to 0$

for simplicity, neglect the non singular term $(1+y)^{-\epsilon}$

Real phase space parametrisation **(momentum mapping)** in terms of Born and radiation variables: $\Phi_R = \Phi_R(\Phi_B, \Phi_{\rm rad})$

$$\int d\Phi_{n+1} = \int d\Phi_n d\Phi_{\rm rad} \sim \int d\Phi_n \tilde{J}(\xi, y, \phi; \Phi_n) \xi^{1-2\epsilon} d\xi (1-y)^{-\epsilon} dy$$

Jacobian of the momentum mapping

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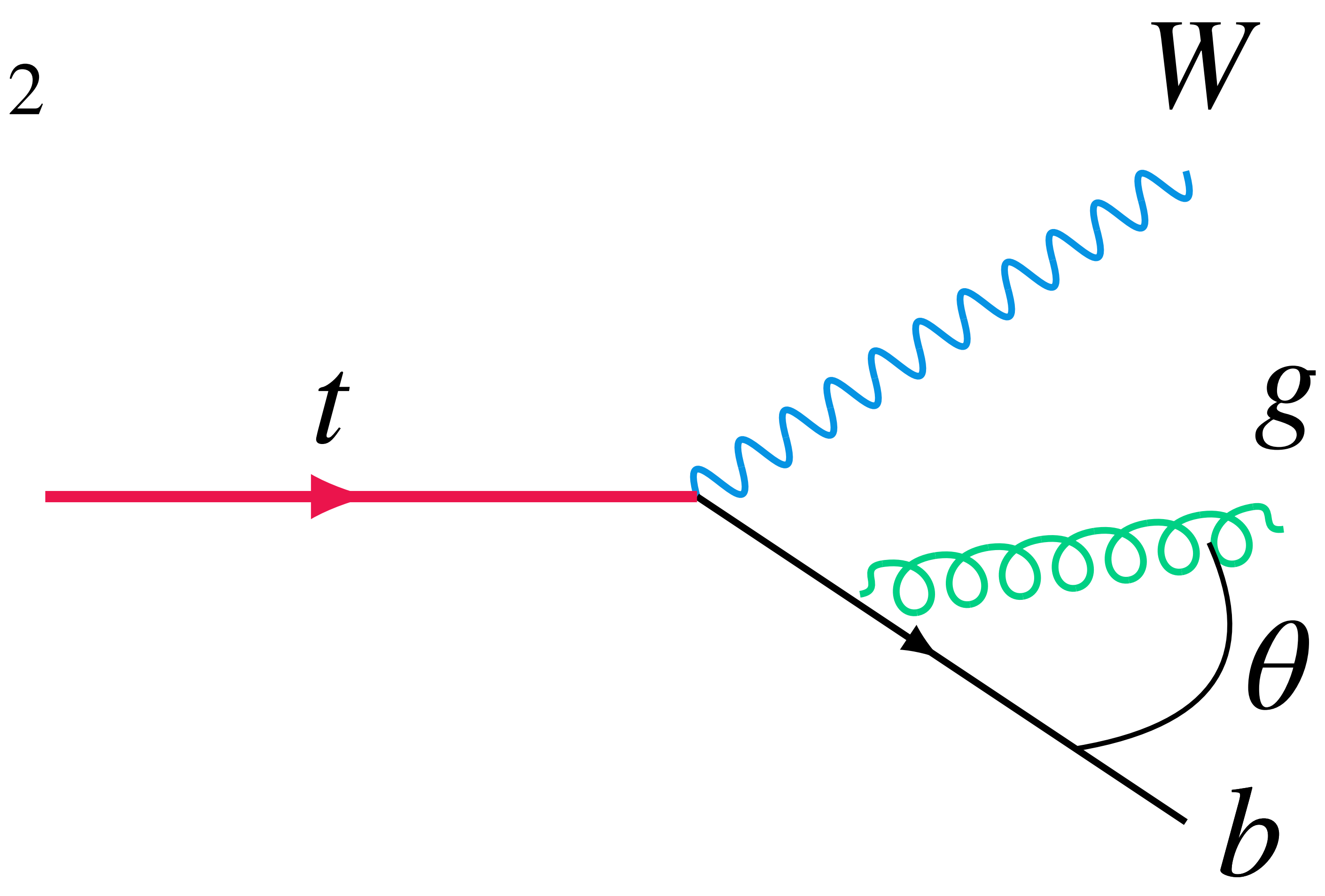

The real matrix element squared behaves in the singular limits as

$$R \sim \frac{1}{\xi^2} \frac{1}{1-y}$$

Then we can rewrite the real emission contribution to the observable as

$$\int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1}) \sim \int d\Phi_n [\xi^2 (1-y) \tilde{J} R F^{n+1}_{\hat{\mathcal{O}}}] \xi^{-1-2\epsilon} d\xi (1-y)^{1-\epsilon} dy$$

with the term in square bracket integrable in four dimensions

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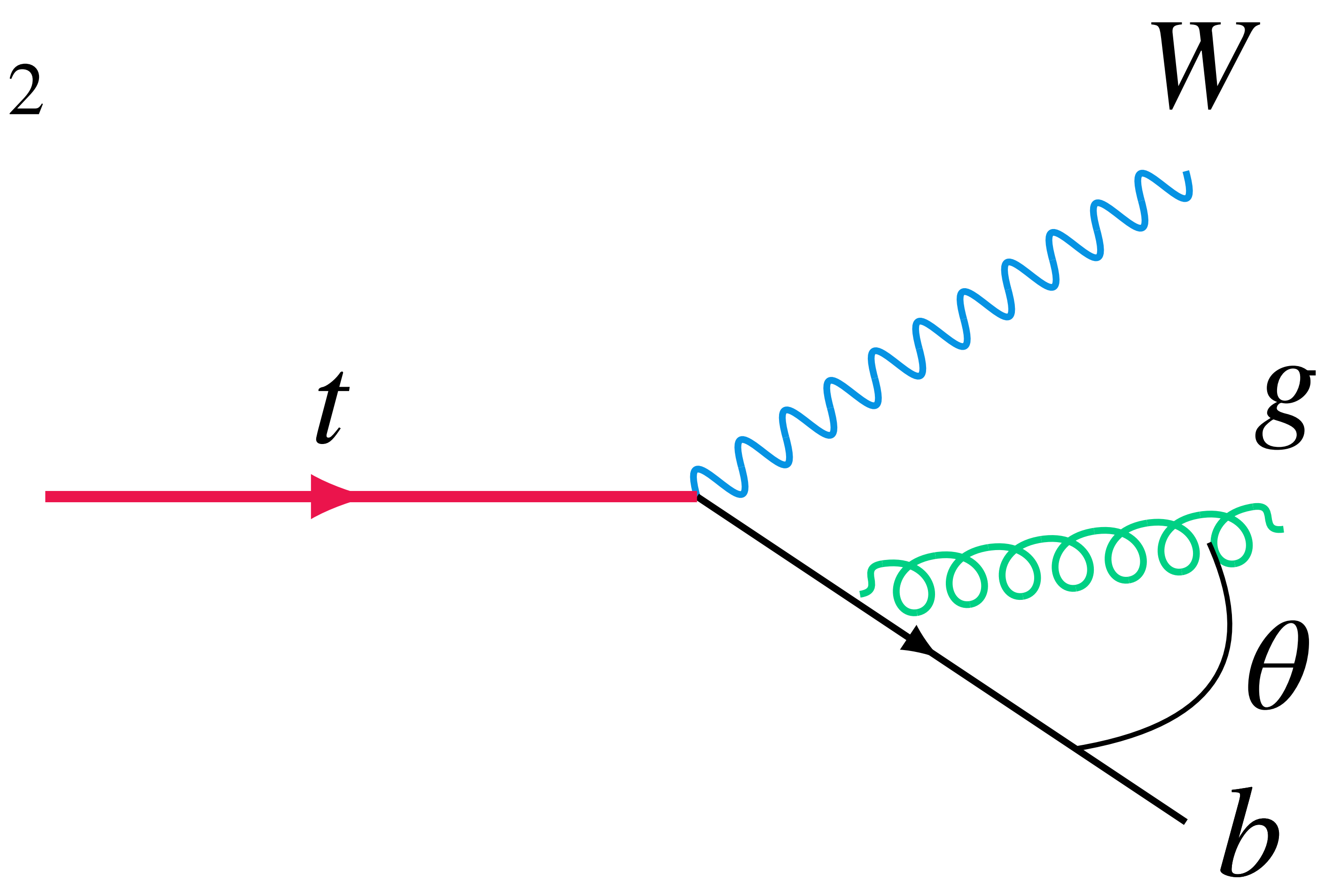
Use the **plus prescription**:
this is achieved by using the following expansions in the space of distributions



$$\xi^{-1-2\epsilon} = -\frac{1}{2\epsilon}\delta(\xi) + \left(\frac{1}{\xi}\right)_+ - 2\epsilon\left(\frac{\ln\xi}{\xi}\right)_+ + \mathcal{O}(\epsilon^2)$$

$$(1-y)^{-1-\epsilon} = -\frac{2^{-\epsilon}}{\epsilon}\delta(1-y) + \left(\frac{1}{1-y}\right)_+ + \mathcal{O}(\epsilon)$$

with the standard definitions (g is a generic test function)

$$\int_0^1 d\xi \left(\frac{1}{\xi}\right)_+ g(\xi) = \int_0^1 d\xi \frac{g(\xi)-g(0)}{\xi}, \quad \int_0^1 d\xi \left(\frac{\ln\xi}{\xi}\right)_+ g(\xi) = \int_0^1 d\xi \frac{g(\xi)-g(0)}{\xi}\ln\xi, \quad \int_{-1}^1 dy \left(\frac{1}{1-y}\right)_+ g(y) = \int_{-1}^1 d\xi \frac{g(y)-g(1)}{1-y},$$

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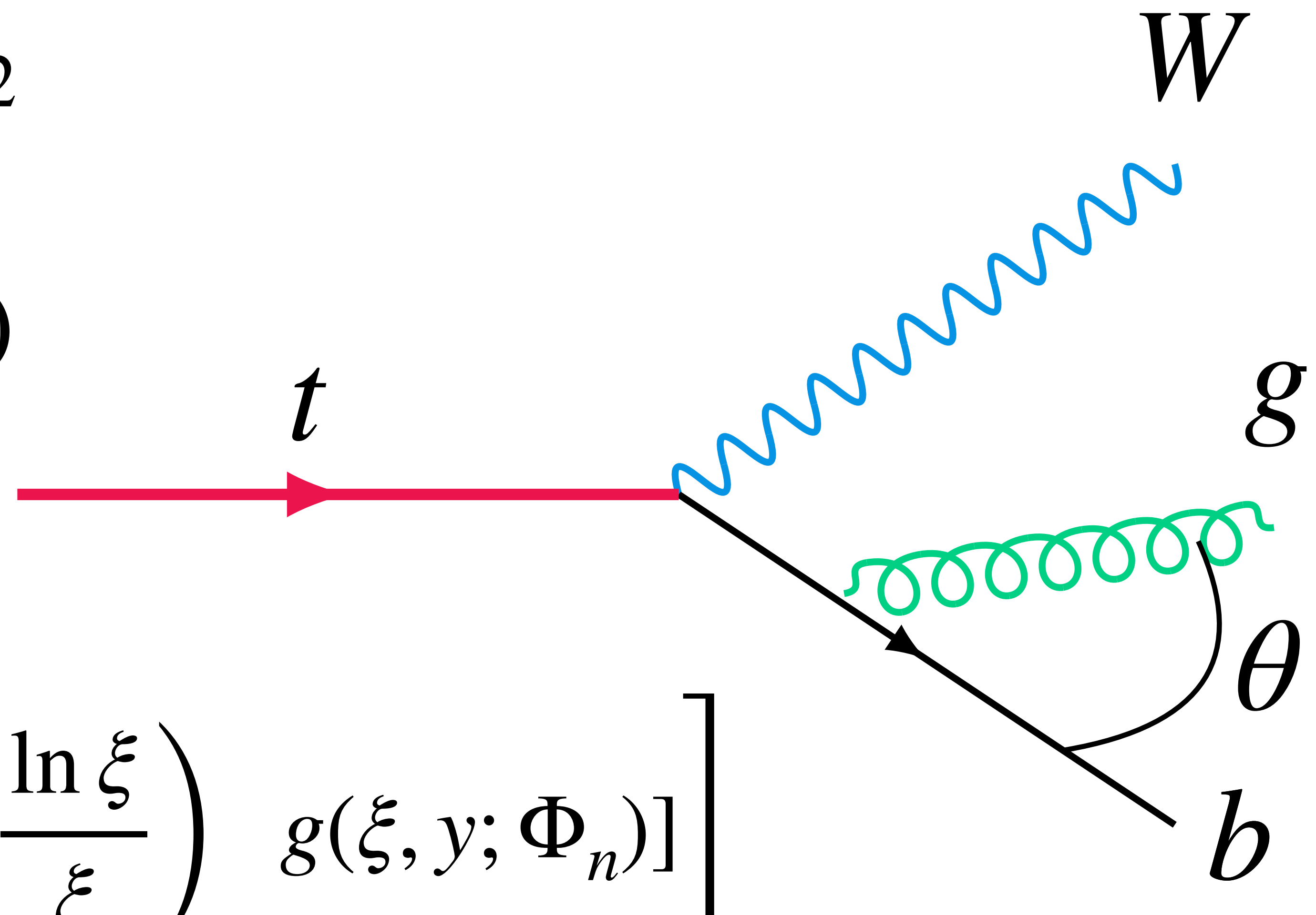

Use the **plus prescription**

$$\int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1}) \sim \int d\Phi_n \int_{-1}^{1} (1-y)^{1-\epsilon} dy \int_0^1 \xi^{-1-2\epsilon} d\xi \overbrace{[\xi^2 (1-y) \tilde{J} R F^{n+1}_{\hat{\mathcal{O}}}]}^{\equiv g(\xi, y; \Phi_n)}$$

$$= \int d\Phi_n \int_{-1}^{1} (1-y)^{1-\epsilon} dy \left[-\frac{1}{2\epsilon} g(0,y;\Phi_n) + \int_0^1 d\xi \left(\frac{1}{\xi} - 2\epsilon \frac{\ln \xi}{\xi} \right)_+ g(\xi, y; \Phi_n)] \right]$$

Integrated counterterms to be combined with the virtual

$$= \int d\Phi_n \left\{ \frac{2^{1-\epsilon}}{\epsilon^2} g(0,1;\Phi_n) - \frac{2^{-\epsilon}}{\epsilon} \int_0^1 d\xi \left(\frac{1}{\xi} - 2\epsilon \frac{\ln \xi}{\xi} \right)_+ g(\xi,1;\Phi_n)] - \frac{1}{2\epsilon} \int_{-1}^{1} dy \left(\frac{1}{1-y} \right)_+ g(0,y;\Phi_n) \right.$$

Finite in four dimensions

$$\left. + \int_{-1}^{1} dy \int_0^1 d\xi \left(\frac{1}{\xi} \right)_+ \left(\frac{1}{1-y} \right)_+ g(\xi, y; \Phi_n) \right\}$$

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$$< \hat{\mathcal{O}} > = \int d\Phi_n [B(\Phi_n) + V(\Phi_n)] F^n_{\hat{\mathcal{O}}}(\Phi_n) + \int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1})$$

$$B = |M_B|^2, V = 2\Re(M_V M_B^*), R = |M_R|^2$$

Use the **plus prescription**

Integrated counterterms

$$\int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1}) \sim \int d\Phi_n \left\{ \frac{2^{1-\epsilon}}{\epsilon^2} g(0,1;\Phi_n) - \frac{2^{-\epsilon}}{\epsilon} \int_0^1 d\xi \left(\frac{1}{\xi} - 2\epsilon \frac{\ln \xi}{\xi} \right)_+ g(\xi,1;\Phi_n)] - \frac{1}{2\epsilon} \int_{-1}^1 dy \left(\frac{1}{1-y} \right)_+ g(0,y;\Phi_n) \right.$$

$$\left. + \int_{-1}^1 dy \int_0^1 d\xi \left(\frac{1}{\xi} \right)_+ \left(\frac{1}{1-y} \right)_+ g(\xi,y;\Phi_n) \right\}$$

Finite in four dimensions

1. Counterterms and overlapping of soft and collinear singularities

$$\int_{-1}^1 dy \int_0^1 d\xi \left(\frac{1}{\xi} \right)_+ \left(\frac{1}{1-y} \right)_+ g(\xi,y;\Phi_n) = \int_{-1}^1 dy \int_0^1 d\xi \frac{g(\xi,y;\Phi_n) - g(0,y;\Phi_n) - g(\xi,1;\Phi_n) + g(0,1;\Phi_n)}{\xi(1-y)}$$

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Finite in four dimensions

2. In the singular limits, no dependence on the IRC measurement function (as in the toy model) and universality thanks to factorisation properties of QCD matrix elements (that can be computed using **eikonal** and **collinear approximations**)

$$\lim_{\xi \to 0} g(\xi, y; \Phi_n) = F^n_{\hat{\mathcal{O}}}(\Phi_n) \tilde{J}(0,y;\Phi_n) \lim_{\xi \to 0} [\xi^2 (1-y) R_s] \qquad \lim_{y \to 1} g(\xi, y; \Phi_n) = F^n_{\hat{\mathcal{O}}}(\Phi_n) \tilde{J}(0,1;\Phi_n) \lim_{\xi \to 1} [\xi^2 (1-y) R_c]$$

Subtraction @ NLO: FKS in two steps (step II)

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The FKS parametrisation works with **one collinear an one soft singularity** at time

ISSUE: how to generalise the construction to more complicated processes ?

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The FKS projection: the art of writing *one* in useful ways (*partition et impera*)

$$1 = \sum_{i \neq j} w_{ij}(\Phi_{n+1}), \qquad R_{ij}(\Phi_{n+1}) \equiv w_{ij}(\Phi_{n+1}) R(\Phi_{n+1})$$

with i, j run over the final-state partons in the $n+1$ phase space. The projector w_{ij} satisfies

- $\lim_{k_i \| k_j} w_{ij} = 1$, $\lim_{E_i \to 0} w_{ij} = 1$ and $\lim_{E_j \to 0} w_{ij} = 1$ (collinear limit of i, j, soft limit of i, soft limit of j)
- smoothly vanishes in all other collinear limits, $\lim_{k_l \| k_m} w_{ij} = 0$ if $(l, m) \neq (i, j)$, and all other soft limits, $\lim_{E_l \to 0} w_{ij} = 0$ if $l \notin \{i, j\}$

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Standard construction of the projectors

- define distances d_{ij} such that $d_{ij} = 0$ if (and only if) $k_i \parallel k_j$. Typically, $d_{ij} = (E_i E_j)^a (1 - \cos\theta_{ij})^b$
- then, define

$$w_{ij} = \frac{1/d_{ij}}{\sum_{l \neq m} 1/d_{lm}}$$

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$$\int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1}) \times 1 = \int d\Phi_{n+1} R(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1}) \times \sum_{i \neq j} w_{ij}(\Phi_{n+1})$$

$$= \sum_{i \neq j} \int d\Phi_{n+1} R_{ij}(\Phi_{n+1}) F^{n+1}_{\hat{\mathcal{O}}}(\Phi_{n+1})$$

Sum of regions with one collinear and one soft singularity at time! Step 1 can be applied for each region

Subtraction @ NLO: FKS in two steps (recap)

STEP I - The plus prescription: FKS parametrisation (**momentum mapping**)

Integrated counterterms

$$\int d\Phi_{n+1} R(\Phi_{n+1}) F_{\hat{\mathcal{O}}}^{n+1}(\Phi_{n+1}) \sim \int d\Phi_n \left\{ \frac{2^{1-\epsilon}}{\epsilon^2} g(0,1;\Phi_n) - \frac{2^{-\epsilon}}{\epsilon} \int_0^1 d\xi \left(\frac{1}{\xi} - 2\epsilon \frac{\ln \xi}{\xi} \right)_+ g(\xi,1;\Phi_n)] - \frac{1}{2\epsilon} \int_{-1}^1 dy \left(\frac{1}{1-y} \right)_+ g(0,y;\Phi_n) \right.$$

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Finite in four dimensions

STEP II - The FKS projection: the art of writing *one* in useful ways (*partition et impera*)

$$\int d\Phi_{n+1} R(\Phi_{n+1}) F_{\hat{\mathcal{O}}}^{n+1}(\Phi_{n+1}) \times 1 = \int d\Phi_{n+1} R(\Phi_{n+1}) F_{\hat{\mathcal{O}}}^{n+1}(\Phi_{n+1}) \times \sum_{i \neq j} w_{ij}(\Phi_{n+1})$$

$$= \sum_{i \neq j} \int d\Phi_{n+1} R_{ij}(\Phi_{n+1}) F_{\hat{\mathcal{O}}}^{n+1}(\Phi_{n+1})$$

Sum of regions with one collinear and one soft singularity at time! Step 1 can be applied for each region

General subtraction algorithm: thanks to factorisation properties of QCD matrix elements in the singular limits, all the necessary (integrated) counterterms can be computed once and for all in a process independent way

Outline

ISSUE: Monte Carlo integration required; how to achieve the cancellation of intermediate singularities while retaining the flexibility of the numerical approach?

@ NLO

- toy-model example
- FKS approach
- CS approach

@NNLO

- anatomy of the complications

Remarks

Subtraction @ NLO: Catani-Seymour like approach

GOAL: design approximants of the real matrix element in d dimensions that

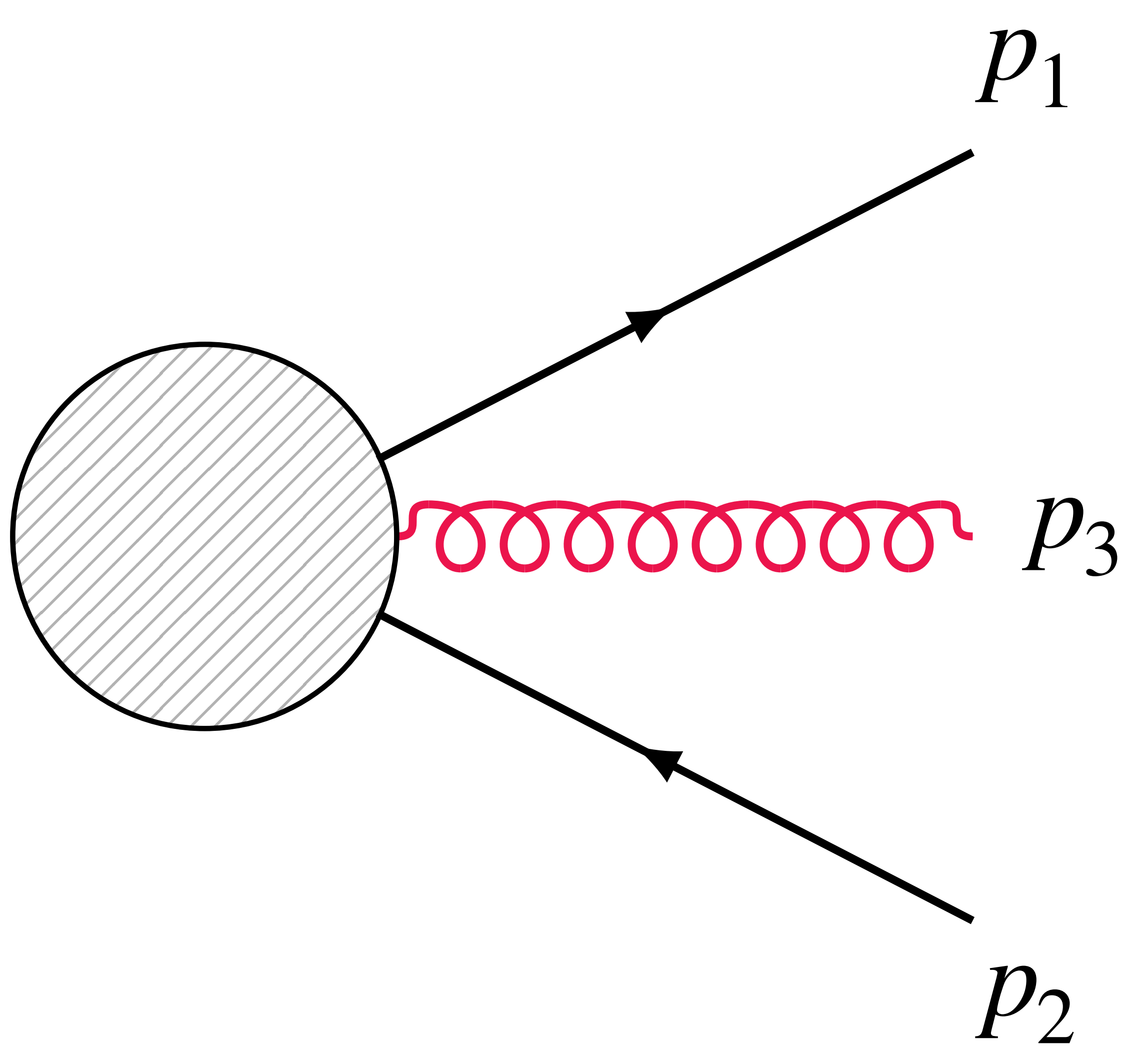
- reproduce the correct singular behaviour in all collinear and soft limits
- are defined in the **entire phase space**
- can be constructed algorithmically
- can be integrated analytically over the d-dimensional 1-particle radiation phase space

IDEA: the singular behaviour of the matrix elements is universal and given by known factorisation formulae

Subtraction @ NLO: Catani-Seymour like approach

Consider a simple example: $\gamma^* \to q(\tilde{p}_1) + \bar{q}(\tilde{p}_2)$

Factorisation of the real matrix element in the relevant limits



a) gluon collinear to the quark: $p_1^\mu = z_1 p^\mu + k_T^\mu - \frac{k_T^2}{z_1}\frac{n^\mu}{2\,p\cdot n}, \quad p_3^\mu = (1-z_1)p^\mu - k_T^\mu - \frac{k_T^2}{1-z_1}\frac{n^\mu}{2\,p\cdot n}$

$$\mathcal{N} = 8\pi\mu^{2\epsilon}\alpha_S C_F \qquad C_{13} = \mathcal{N}\frac{1}{2\,p_1\cdot p_3}\left[\frac{1+z_1^2}{1-z_1} - \epsilon(1-z_1)\right]|M_{\gamma^*\to q\bar{q}}(p_1+p_3, p_2)|^2$$

b) gluon collinear to the antiquark: $p_2^\mu = z_2 p^\mu + k_T^\mu - \frac{k_T^2}{z_2}\frac{n^\mu}{2\,p\cdot n}, \quad p_3^\mu = (1-z_2)p^\mu - k_T^\mu - \frac{k_T^2}{1-z_2}\frac{n^\mu}{2\,p\cdot n}$

$$C_{23} = \mathcal{N}\frac{1}{2\,p_2\cdot p_3}\left[\frac{1+z_2^2}{1-z_2} - \epsilon(1-z_2)\right]|M_{\gamma^*\to q\bar{q}}(p_1, p_2+p_3)|^2$$

c) soft gluon: $p_3 \to 0$

$$S_3 = \mathcal{N}\frac{p_1\cdot p_2}{p_1\cdot p_3\; p_2\cdot p_3}|M_{\gamma^*\to q\bar{q}}(p_1, p_2)|^2$$

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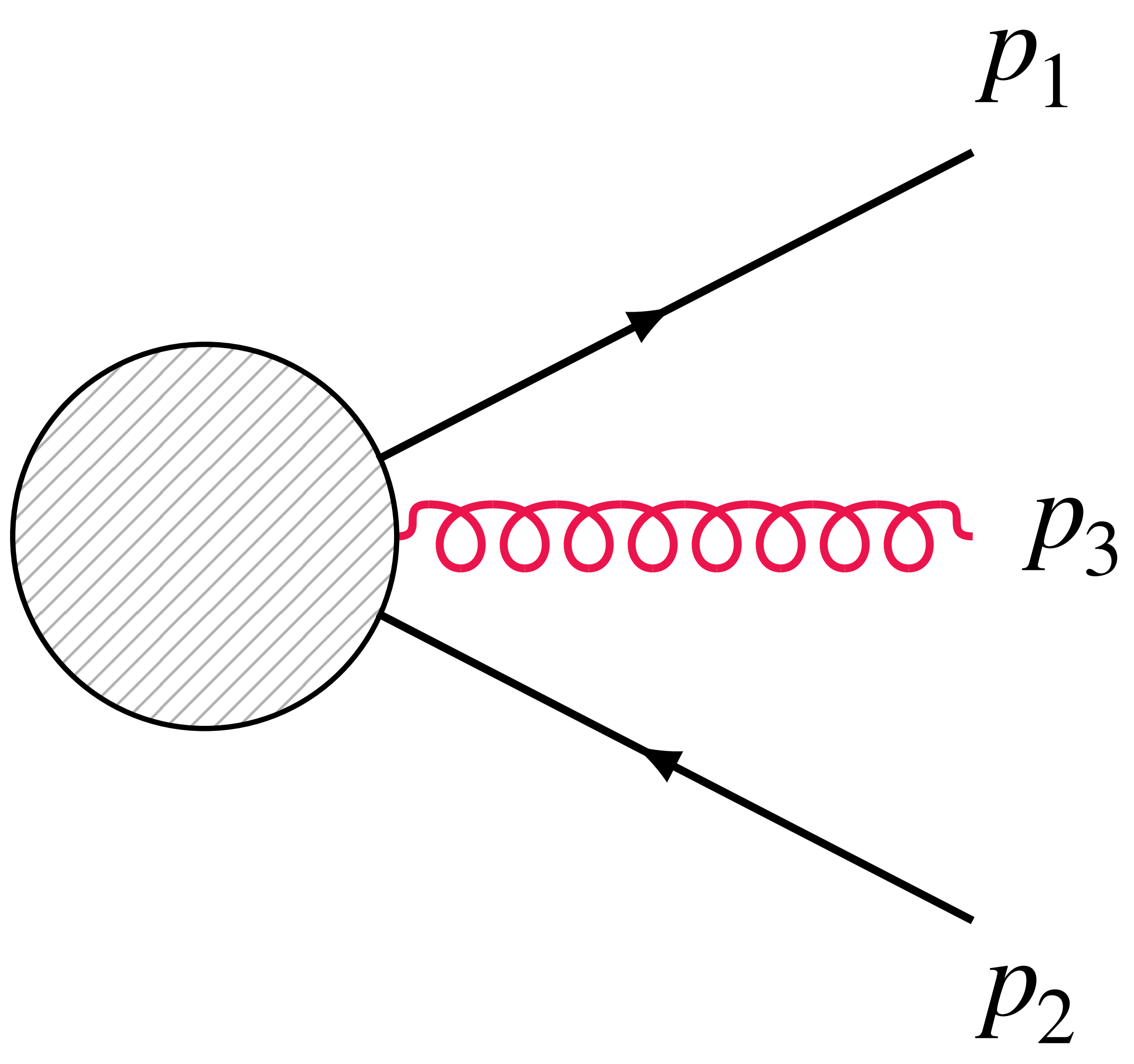
Factorisation of the real matrix element in the relevant limits



It is tempting to write the approximant as

$$A_1 = C_{13} + C_{23} + S_3$$

but

1. the formula is incorrect in the simultaneous soft and collinear limits because of double counting **(overlapping singularities)**
2. the expressions C_{13}, C_{23} and S_3 **cannot be evaluated away from their corresponding singular regions** as momentum conservation and mass shell conditions are not satisfied and collinear fractions $z_{1,2}$ are not well defined

Solutions given by

1. **matching**
2. **extension**

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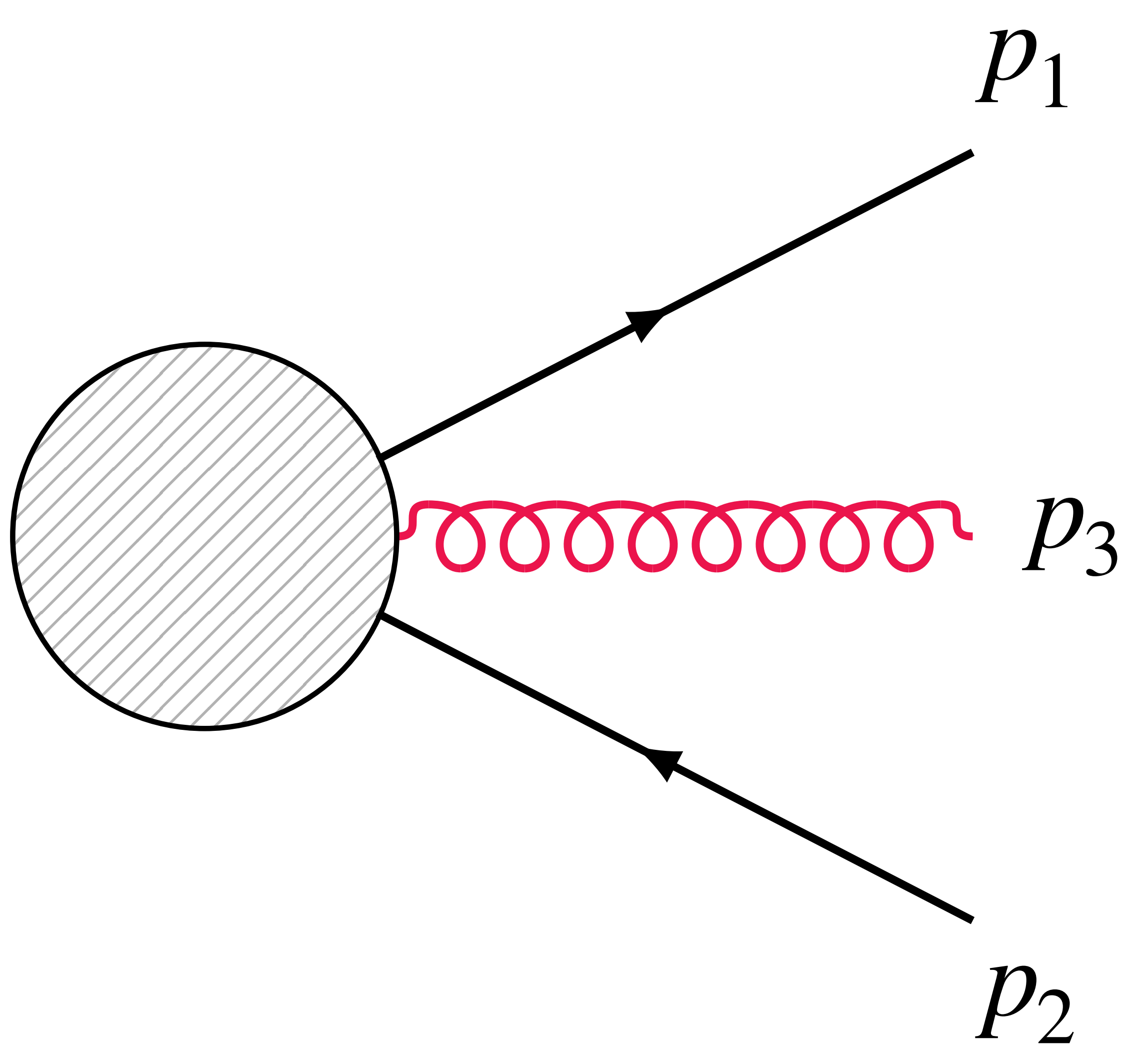


1. **Matching** (analogously in the limits collinear to the anti quark)

$$\lim_{p_3\to 0} C_{13} = \mathcal{N} \frac{1}{2\,p_1\cdot p_3}\frac{2}{1-z_1}\,|M_{\gamma^*\to q\bar{q}}(p1,p2)|^2 \qquad p_3 \to 0 \sim z_1 \to 1$$

$$\lim_{p_1\|p_3} S_3 = \mathcal{N} \frac{z_1 \cancel{p\cdot p_2}}{p_1\cdot p_3\,(1-z_1)\,\cancel{p\cdot p_2}}\,|M_{\gamma^*\to q\bar{q}}(p1,p2)|^2$$

$$= \mathcal{N} \frac{1}{2\,p_1\cdot p_3}\frac{2z_1}{(1-z_1)}\,|M_{\gamma^*\to q\bar{q}}(p1,p2)|^2 \equiv C_{13}S_3$$

$$p_1 \parallel p_3 : p_1 \sim z_1 p, p_3 \sim (1-z_1)p$$

$$\lim_{p_1\|p_3} (S_3 - C_{13}S_3) = 0 \qquad \text{by definition}$$

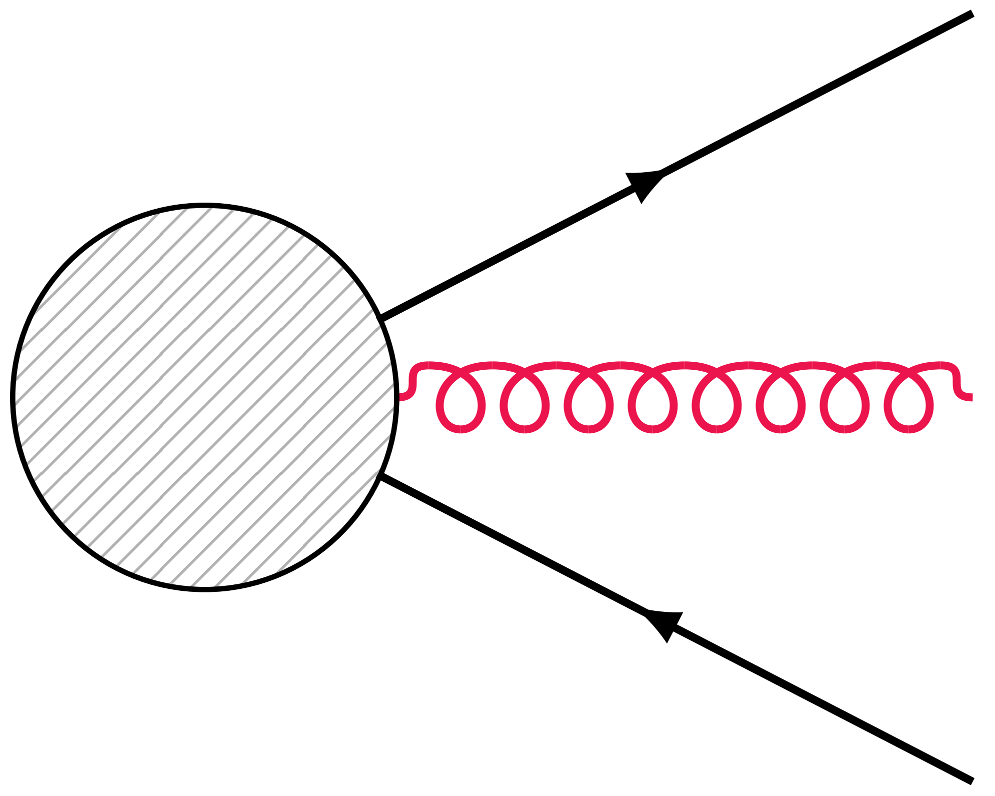
$$\lim_{p_3\to 0} (C_{13} - C_{13}S_3) = \mathcal{N}\frac{1}{2\,p_1\cdot p_3}\,|M_{\gamma^*\to q\bar{q}}(p1,p2)|^2 \lim_{z_1\to 1}\left[\frac{2}{1-z_1} - \frac{2z_1}{1-z_1}\right] = 0$$

$$A_1 = C_{13} + C_{23} + S_3 - C_{13}S_3 - C_{23}S_3$$

Subtraction @ NLO: Catani-Seymour like approach

Consider a simple example: $\gamma^* \to q(\tilde{p}_1) + \bar{q}(\tilde{p}_2)$

Factorisation of the real matrix element in the relevant limits



1. Notice that defining instead

$$\lim_{p_3 \to 0} C_{13} = \mathcal{N} \frac{1}{2\,p_1 \cdot p_3} \frac{2}{1 - z_1} |M_{\gamma^* \to q\bar{q}}(p1,p2)|^2 = S_3 C_{13}$$

$$\lim_{p_1 \| p_3} S_3 = \mathcal{N} \frac{z_1 \cancel{p \cdot p_2}}{p_1 \cdot p_3\,(1 - z_1)\,\cancel{p \cdot p_2}} |M_{\gamma^* \to q\bar{q}}(p1,p2)|^2$$

$$= \mathcal{N} \frac{1}{2\,p_1 \cdot p_3} \frac{2z_1}{(1 - z_1)} |M_{\gamma^* \to q\bar{q}}(p1,p2)|^2$$

$$\lim_{p_3 \to 0} (C_{13} - S_3 C_{13}) = 0 \quad \text{by definition}$$

$$\lim_{p_1 \| p_3} (S_3 - S_3 C_{13}) = \mathcal{N} \frac{1}{2\,p_1 \cdot p_3} |M_{\gamma^* \to q\bar{q}}(p1,p2)|^2 \left[\frac{2z_1}{1 - z_1} - \frac{2}{1 - z_1} \right] \neq 0$$

✘

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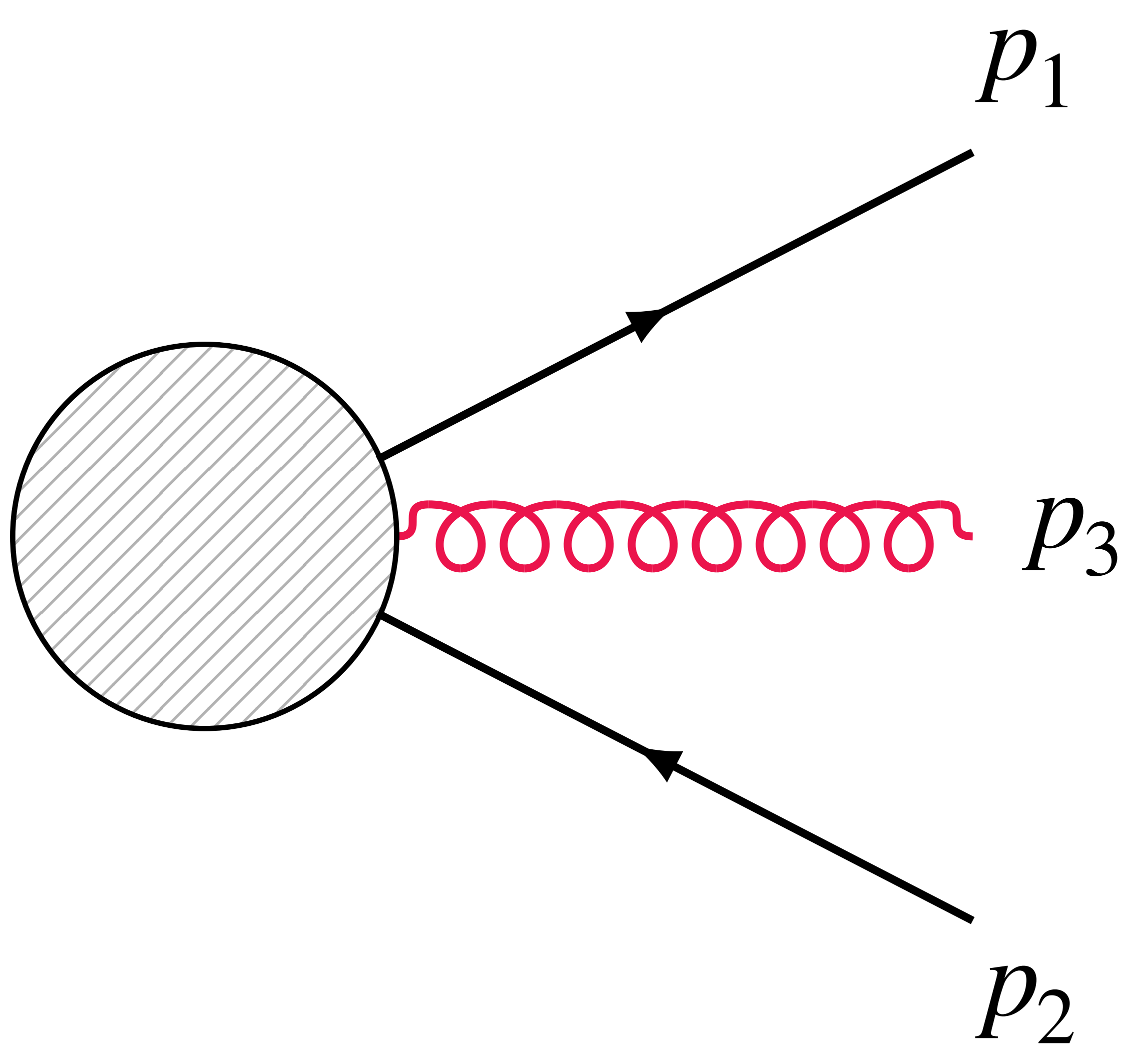
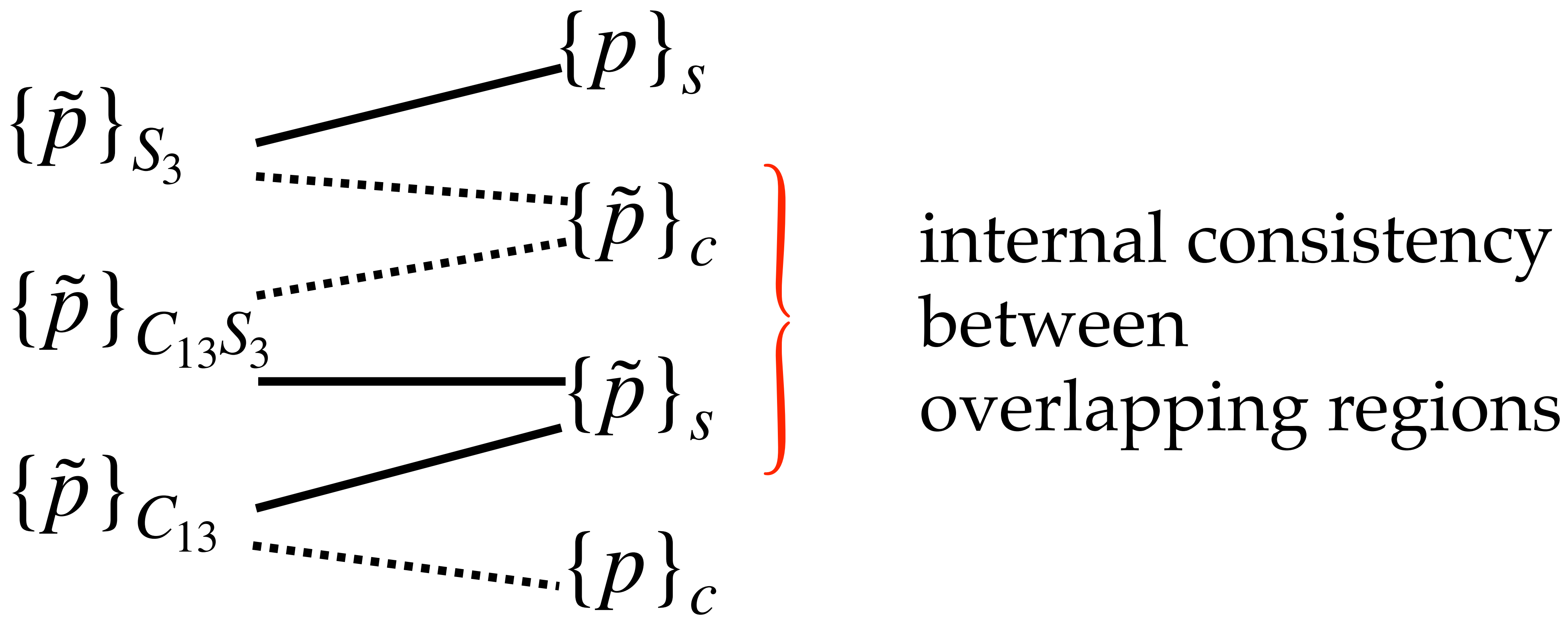
Factorisation of the real matrix element in the relevant limits



2. **Extension requires**

Momentum mappings from real to Born momenta for the evaluation of the reduced Born Matrix element. They must have the following properties

- ensure momentum conservation and mass shell of all particles
- recover the expected behaviour in the corresponding singular limit
- (lead to exact factorisation of the phase space)



Similarly, one needs to consistently extend the definition of **collinear fractions** $z_{1,2}$

Subtraction @ NLO: Catani-Seymour

[Catani, Seymour (1998)]

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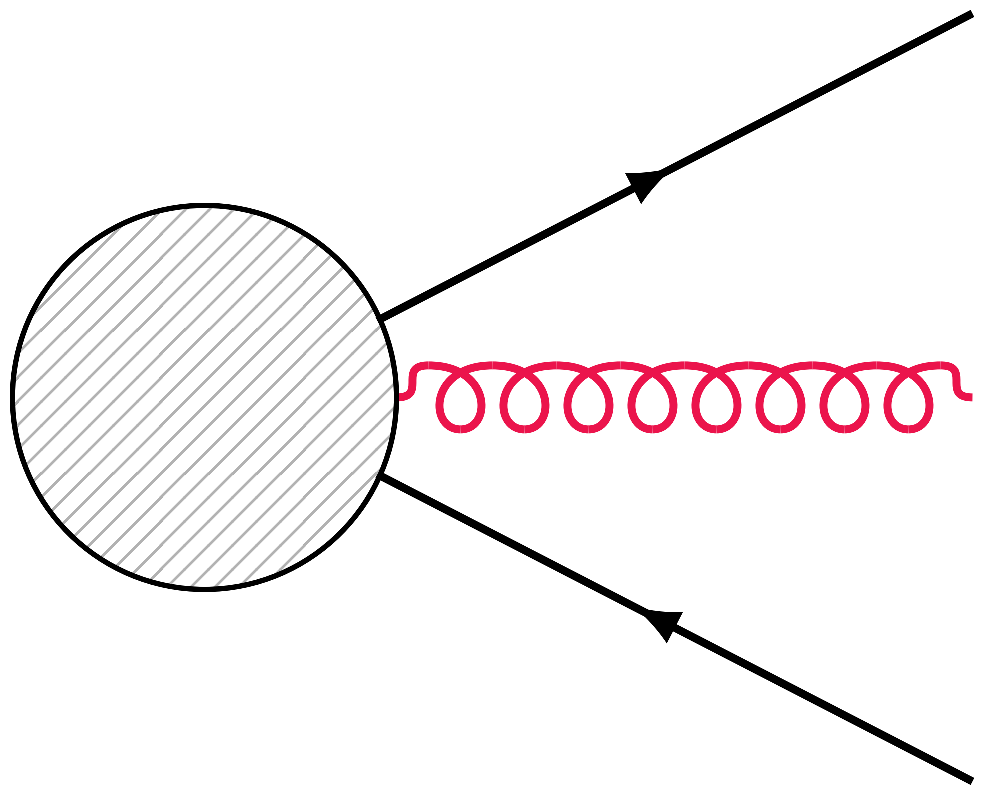

CATANI-SEYMOUR DIPOLES (no hadrons in the initial-state)

The approximant is written as a sum of *dipoles*

$$A_1 = V_{13,2}(p_1, p_2, p_3)\,|\,M_{\gamma^* \to q\bar{q}}(\tilde{p}_1, \tilde{p}_2)\,|^2 + V_{23,1}(p_1, p_2, p_3)\,|\,M_{\gamma^* \to q\bar{q}}(\tilde{p}'_1, \tilde{p}'_2)\,|^2$$

A dipole $V_{ij,k}$ include a pair (i, j), interpreted as coming from a splitting process $\tilde{ij} \to i + j$, and a "spectator" parton that absorbs the recoil of the splitting and ensures the correct treatment of colour and spin correlations (*trivial in the considered example*)

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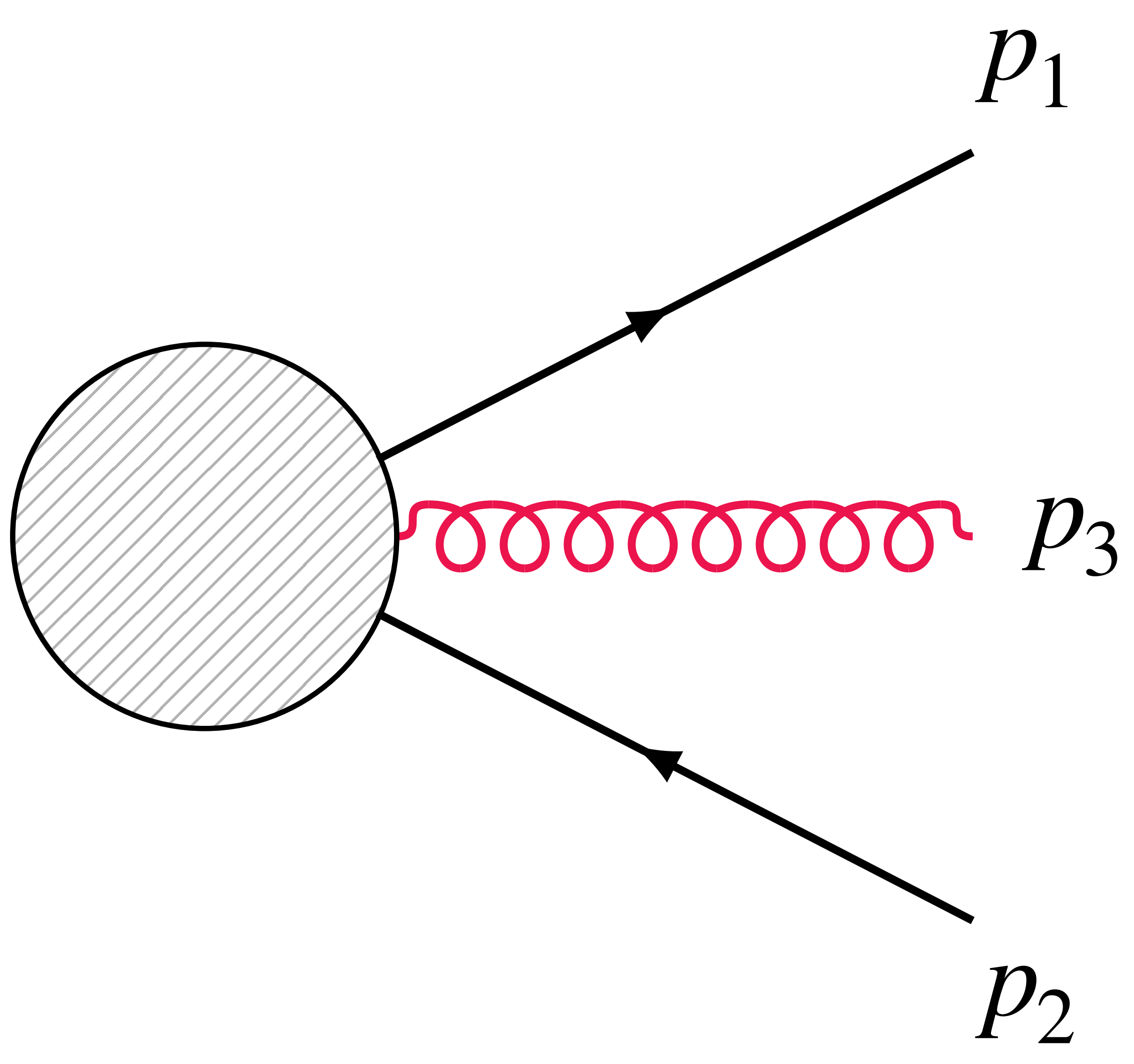

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Momentum mapping: $\{p_i, p_j, p_k\} \to \{\tilde{p}_{ij}, \tilde{p}_k\}$

momentum conservation

$$\tilde{p}^\mu_{ij} + \tilde{p}^\mu_k = p^\mu_i + p^\mu_j + p^\mu_k$$

$$\tilde{p}^\mu_{ij} = p^\mu_i + p^\mu_j + \alpha p^\mu_k$$

$$\tilde{p}^\mu_k = (1-\alpha) p^\mu_k$$

mass-shell relation

$$\tilde{p}^2_{ij} = 0 \implies \alpha = -\frac{p_i \cdot p_j}{(p_i + p_j) \cdot p_k}$$

Usual α is replaced by $y_{ij,k} = -\frac{\alpha}{1-\alpha} = \frac{p_i \cdot p_j}{p_i \cdot p_j + p_i \cdot p_k p +_j \cdot p_k}$

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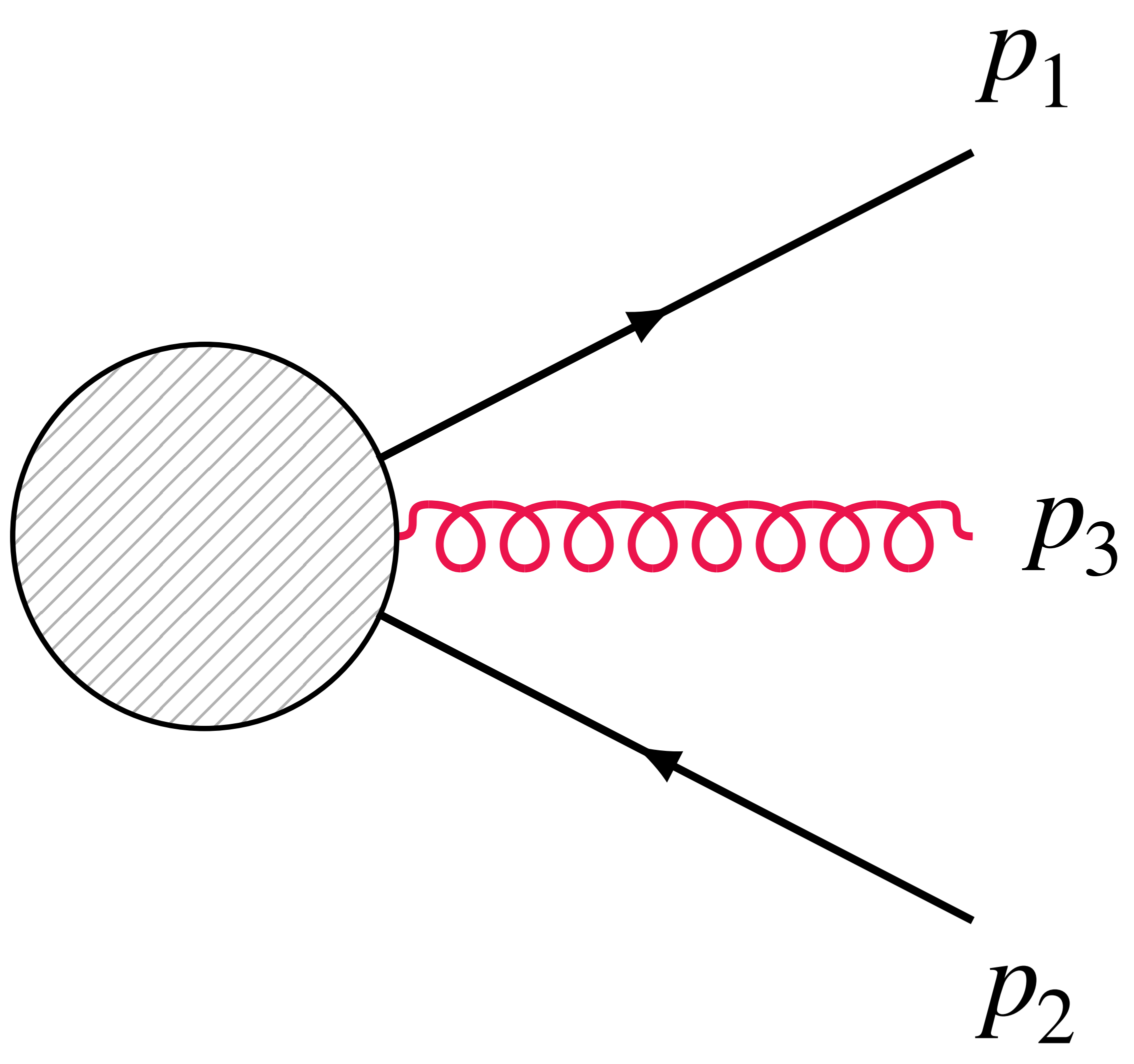

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Momentum mapping: $\{p_i, p_j, p_k\} \to \{\tilde{p}_{ij}, \tilde{p}_k\}$

$$\tilde{p}^\mu_{ij} = p^\mu_i + p^\mu_j - \frac{y_{ij,k}}{1 - y_{ij,k}} p^\mu_k \qquad y_{ij,k} = \frac{p_i \cdot p_j}{p_i \cdot p_j + p_i \cdot p_k p +_j \cdot p_k}$$

$$\tilde{p}^\mu_k = \frac{1}{1 - y_{ij,k}} p^\mu_k$$

In the relevant soft/collinear limit $p_i \cdot p_j \to 0$, $y \sim 0$ and then, as expected,

$$\tilde{p}^\mu_{ij} \sim p^\mu_i + p^\mu_j, \quad \tilde{p}^\mu_k \sim \tilde{p}^\mu_k$$

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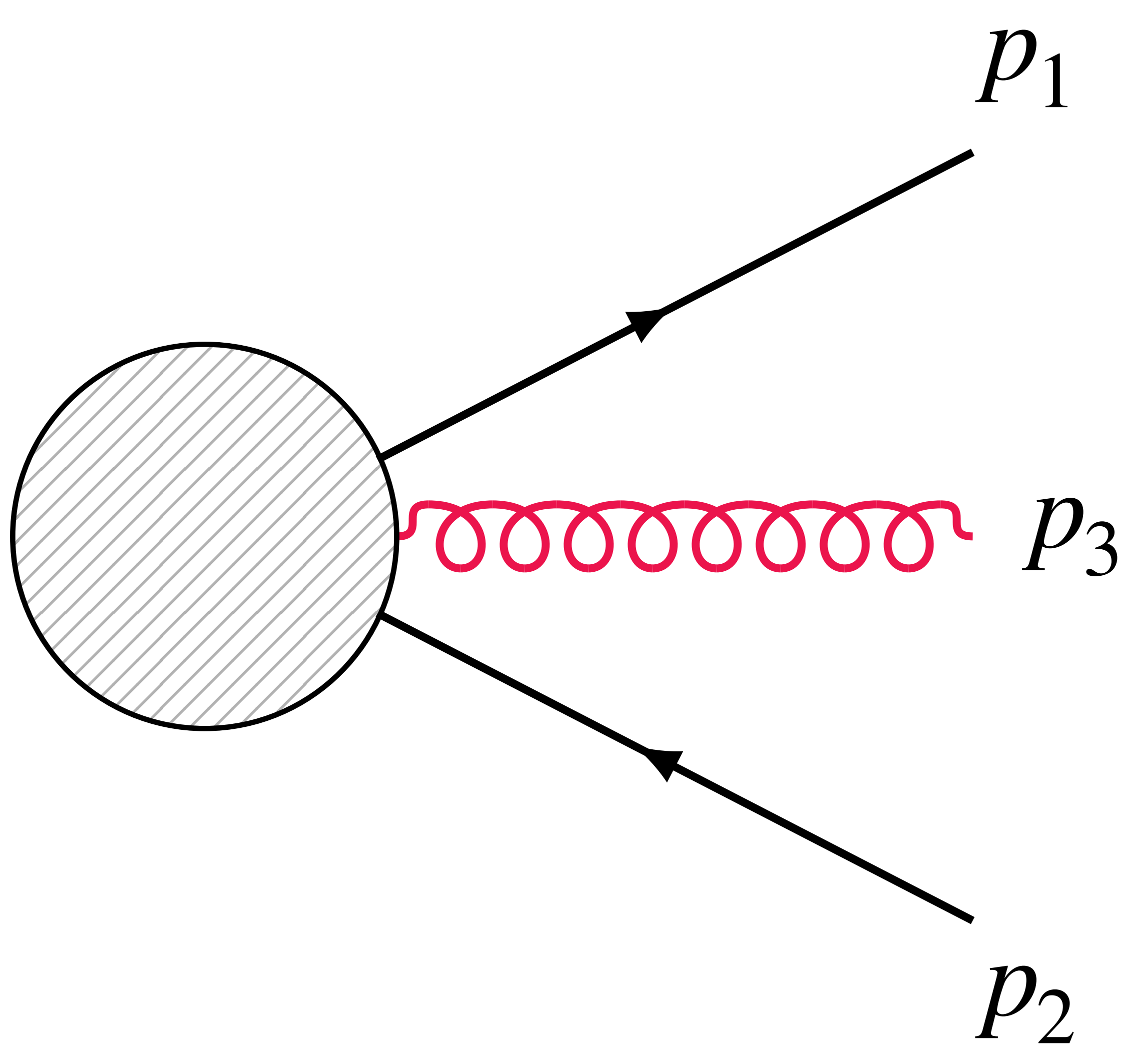

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Momentum mapping: $\{p_i, p_j, p_k\} \to \{\tilde{p}_{ij}, \tilde{p}_k\}$

$$\tilde{p}^\mu_{ij} = p^\mu_i + p^\mu_j - \frac{y_{ij,k}}{1-y_{ij,k}} p^\mu_k \qquad y_{ij,k} = \frac{p_i \cdot p_j}{p_i \cdot p_j + p_i \cdot p_k p +_j \cdot p_k}$$

$$\tilde{p}^\mu_k = \frac{1}{1-y_{ij,k}} p^\mu_k$$

collinear limit

$$z_i = \frac{p_i \cdot p_k}{(p_i + p_j)\cdot p_k} = \frac{p_i \cdot \tilde{p}_k}{\tilde{p}_{ij}\cdot \tilde{p}_k} \qquad z_i = \frac{p_i \cdot p_k}{(p_i+p_j)\cdot p_k} \to z_i^c \frac{\cancel{p \cdot p_k}}{\cancel{p \cdot p_k}} = z_i^c$$

soft limit

$$z_i \to 1$$

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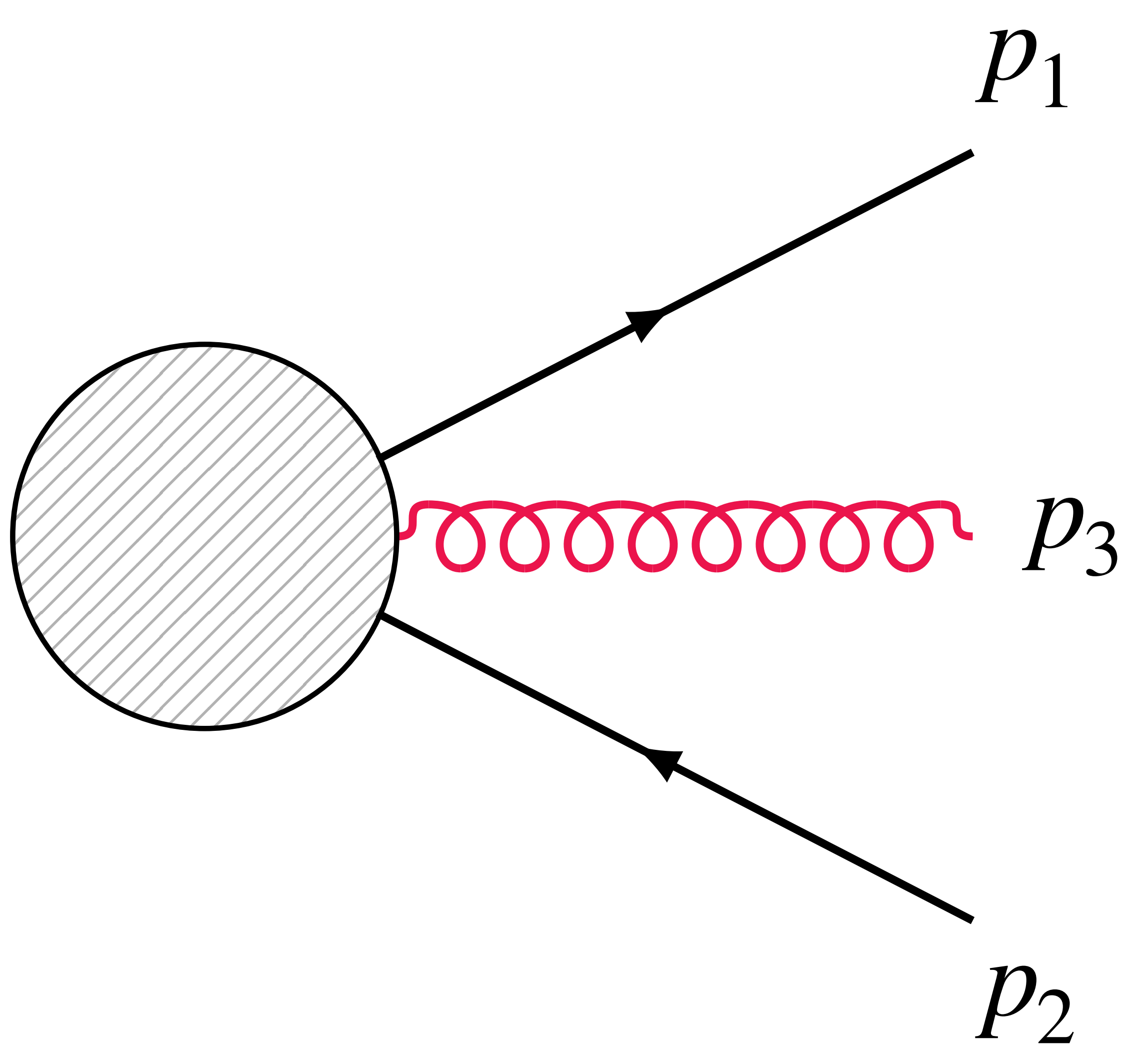


CATANI-SEYMOUR DIPOLES (no hadrons in the initial-state)

The approximant is written as a sum of *dipoles*

$$A_1 = V_{13,2}(p_1,p_2,p_3)\,|M_{\gamma^*\to q\bar{q}}(\tilde{p}_1,\tilde{p}_2)|^2 + V_{23,1}(p_1,p_2,p_3)\,|M_{\gamma^*\to q\bar{q}}(\tilde{p}'_1,\tilde{p}'_2)|^2$$

Dipole functions: start from **eikonal approximation** and **apply partial fractioning**

$$S_{ijk} = \mathcal{N}\frac{p_i\cdot p_k}{p_i\cdot p_j\; p_k\cdot p_j}|M_{\gamma^*\to q\bar{q}}|^2 = \mathcal{N}\left[\frac{p_i\cdot p_k}{p_i\cdot p_j\;(p_i+p_k)\cdot p_j} + \frac{p_i\cdot p_k}{(p_i+p_k)\cdot p_j\; p_k\cdot p_j}\right]|M_{\gamma^*\to q\bar{q}}|^2$$

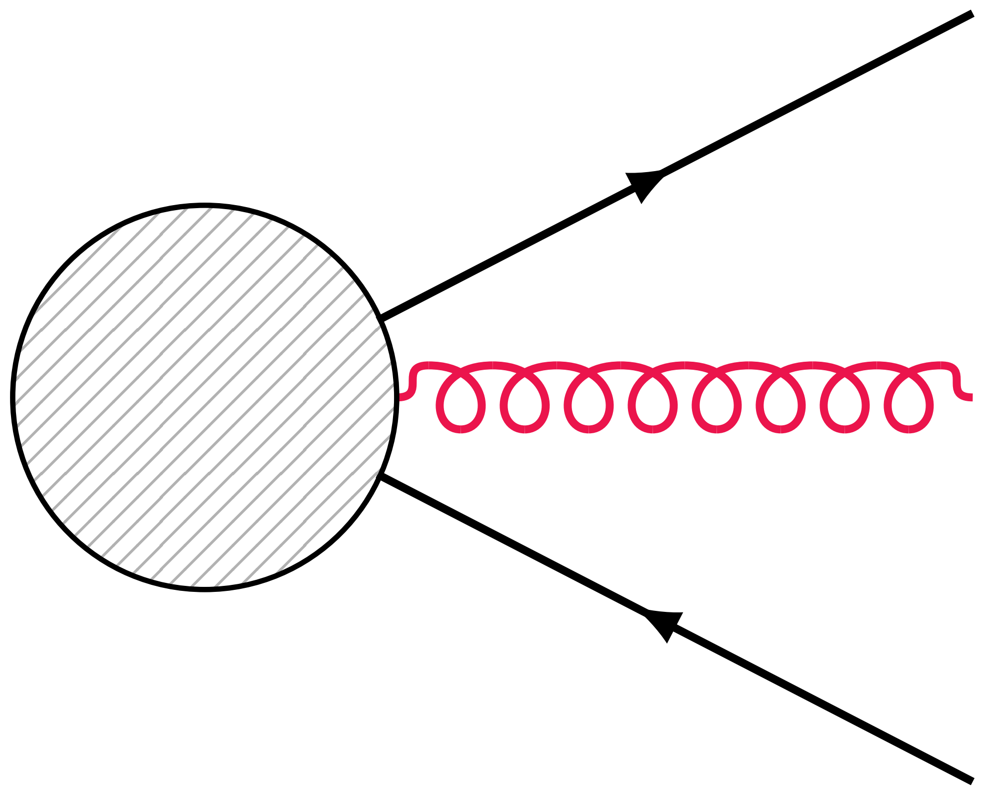
$S_{ij,k}$: only collinear to p_i; contributes to $V_{ij,k}$

$S_{kj,i}$: only collinear to p_k; contributes to $V_{kj,i}$

Subtraction @ NLO: Catani-Seymour

[Catani, Seymour (1998)]

Consider a simple example: $\gamma^* \to q(\tilde{p}_1) + \bar{q}(\tilde{p}_2)$



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Dipole functions: match C_{ij} and $S_{ij,k}$ (smooth interpolation)

$$C_{ij} = \mathcal{N}\frac{1}{2\,p_i\cdot p_j}\left[\frac{1+z_i^2}{1-z_i} - \epsilon(1-z_i)\right]|M_{\gamma^*\to q\bar{q}}|^2$$

$$S_{ij,k} = \mathcal{N}\frac{p_i\cdot p_k}{p_i\cdot p_j\,(p_i+p_k)\cdot p_j}|M_{\gamma^*\to q\bar{q}}|^2 = \mathcal{N}\frac{1}{2\,p_i\cdot p_j}\frac{2(1-y_{ij,k})z_i}{1-z_i(1-y_{ij,k})}|M_{\gamma^*\to q\bar{q}}|^2$$

$$V_{ij,k} = C_{ij} + S_{ij,k} - C_{ij}S_{ij,k} = \mathcal{N}\frac{1}{2\,p_i\cdot p_j}\left[\frac{1+z_i^2}{1-z_i} - \epsilon(1-z_i) + \frac{2(1-y_{ij,k})z_i}{1-z_i(1-y_{ij,k})} - \frac{2z_i}{1-z_i}\right]|M_{\gamma^*\to q\bar{q}}|^2$$

$$= \mathcal{N}\frac{1}{2\,p_i\cdot p_j}\left[\frac{2}{1-z_i(1-y_{ij,k})} - (1+z_i) - \epsilon(1-z_i)\right]|M_{\gamma^*\to q\bar{q}}|^2$$

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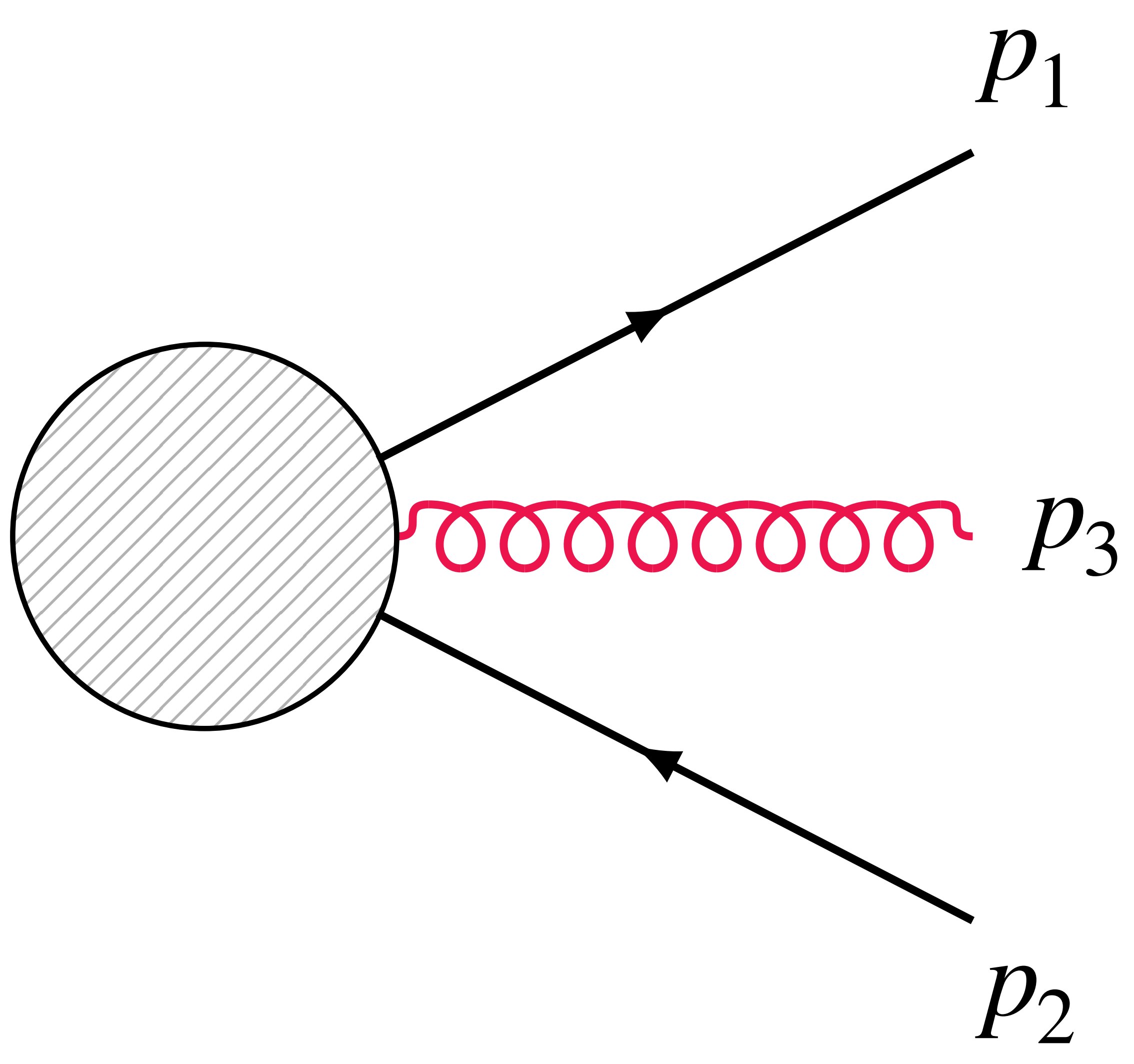

CATANI-SEYMOUR DIPOLES (no hadrons in the initial-state)

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Integrated counterterm:

exact factorisation $\quad d\Phi(p_i,p_j,p_k;q) = d\Phi(\tilde{p}_{ij},\tilde{p}_k;q)\,d\Phi_{\text{rad}}(\tilde{p}_{ij},\tilde{p}_k)$

$$d\Phi_{\text{rad}}(\tilde{p}_{ij},\tilde{p}_k) = \frac{(2\tilde{p}_{ij}\cdot\tilde{p}_k)^{1-\epsilon}}{16\pi^2}\frac{d\Omega^{d-2}}{(2\pi)^{1-2\epsilon}}dz_i dy_{ij,k}\Theta(z_i(1-z_i))\Theta(y_{ij,k}(1-y_{ij,k}))$$
$$\times\,(z_i(1-z_i))^{-\epsilon}(1-y_{ij,k})^{1-2\epsilon}y_{ij,k}^{-\epsilon}$$

$$\mathscr{V}_{ij,k} = \int d\Phi_{\text{rad}}(\tilde{p}_{ij},\tilde{p}_k)V_{ij,k} = \frac{\alpha_S}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{2\tilde{p}_{ij}\cdot\tilde{p}_k}\right)^{\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}C_F\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\frac{3+\epsilon}{2(1-3\epsilon)}\right]$$

Subtraction @ NLO: NLO revolution!

FKS subtraction

- momentum recoil distributed among all particles (global) **complexity** scales as $n \times (n-1) \times 1 \sim n^2$
- construction starts from **collinear** radiation
- **general algorithm**
- automated in different (public) programs: POWHEG BOX, MadGraph5_aMC@NLO …

CS dipole subtraction

- momentum recoil absorbed by one particle **numerical complexity** scales as $n \times (n-1) \times (n-2) \sim n^3$
- construction starts from **soft** radiation
- **general algorithm**
- automated in different (public) programs: Sherpa, Helac-NLO, MadDipole, Matrix …

technicalities not covered in this talk (together with identified incoming hadrons)

Numerical evaluation of **tree-level** (including **colour- and spin-correlated**) and **1-loop** QCD (and EW and BSM) virtual amplitudes automated in different public generators: OpenLoops, Recola, GoSam, MadLoop, NLOX …

Complete automation: NLO QCD (and EW) corrections to any *desirable* processes for LHC physics can be computed by pressing a button

Outline

ISSUE: Monte Carlo integration required; how to achieve the cancellation of intermediate singularities while retaining the flexibility of the numerical approach?

@ NLO

- toy-model example
- FKS approach
- CS approach

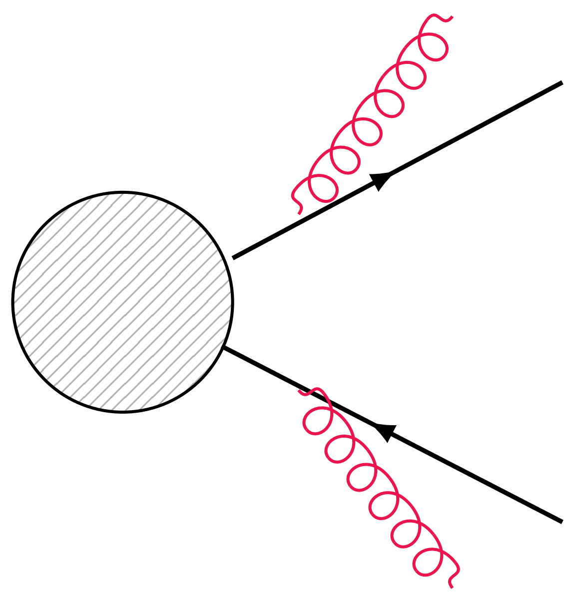
@NNLO

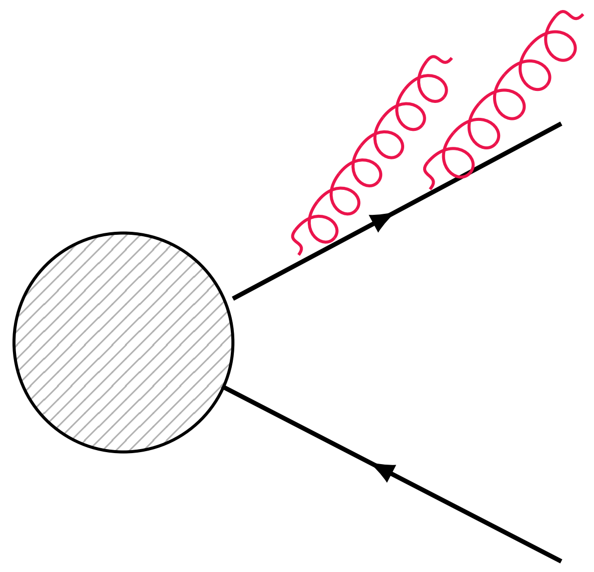
- anatomy of the complications

Remarks

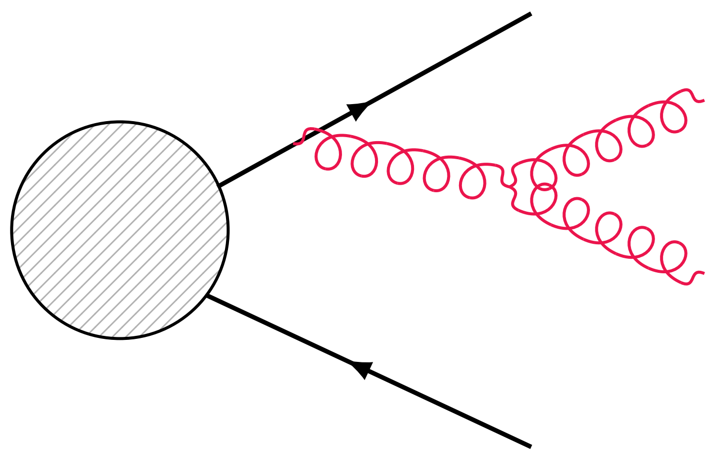
Subtraction @ NNLO: anatomy of "complications"

Double real: more involved structure of singular limits. Overlapping of singularities is a more severe problem





double collinear limit



triple collinear limit

Subtraction @ NNLO: anatomy of "complications"

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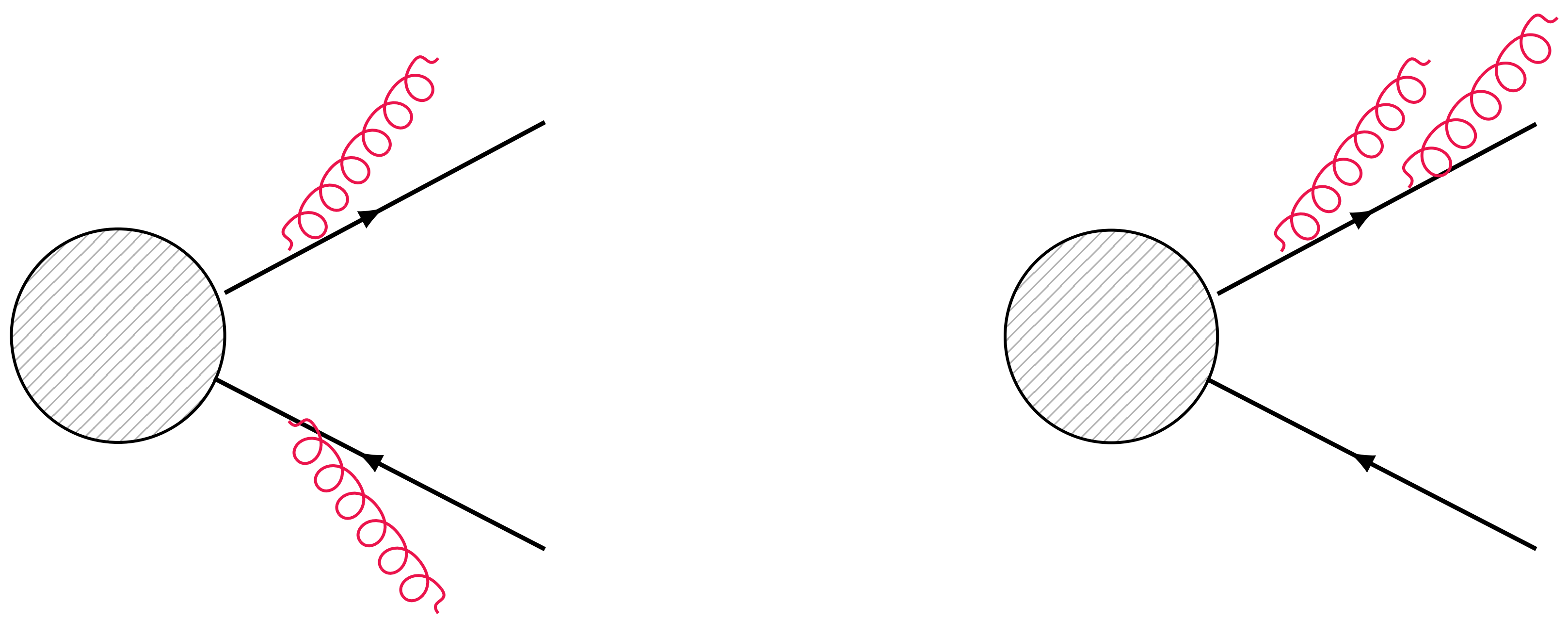
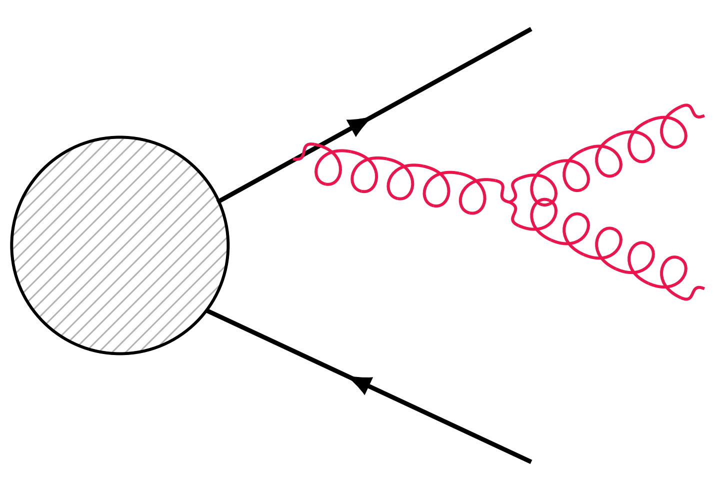


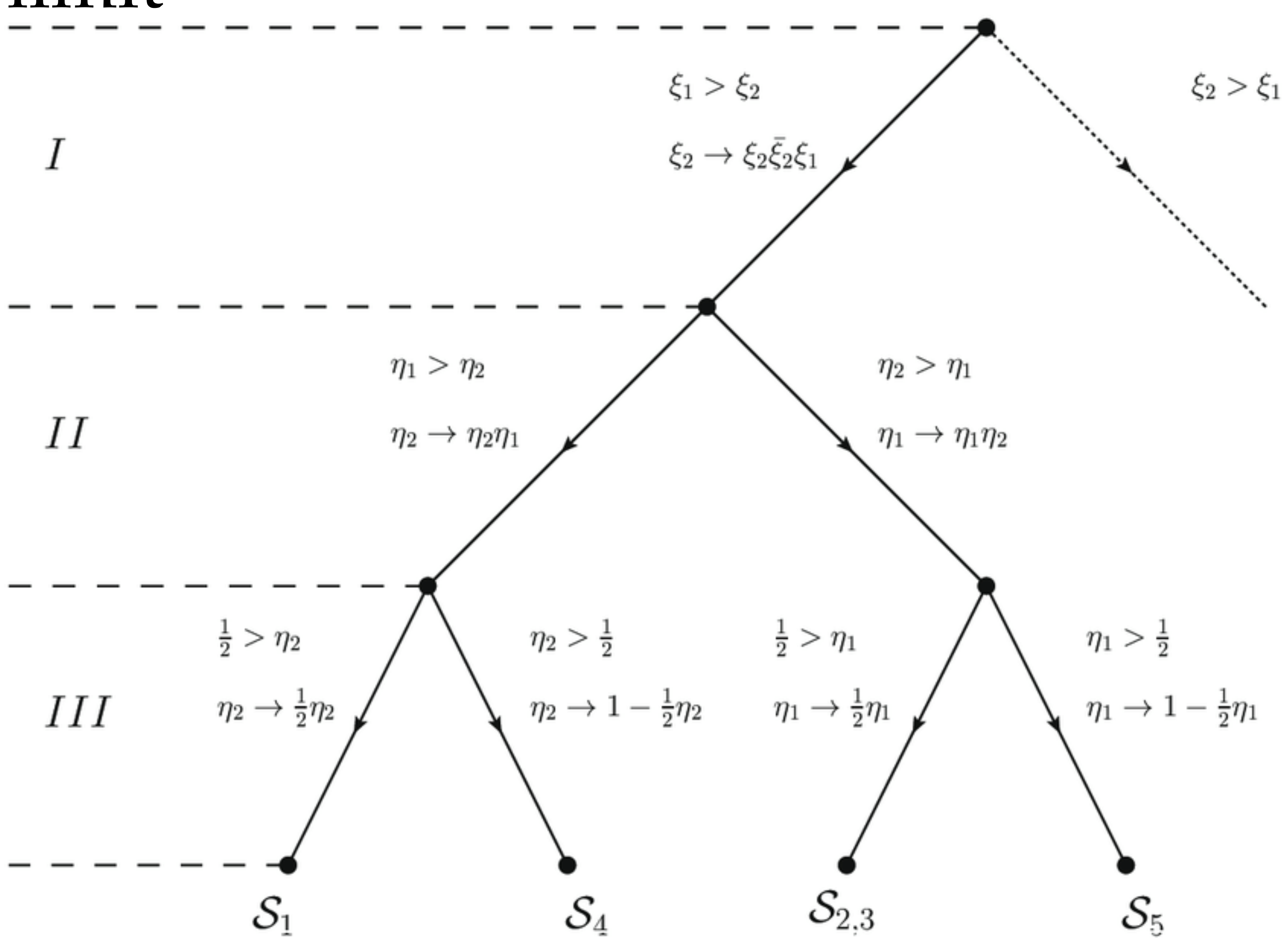
double collinear limit



triple collinear limit

$$\hat{\eta}_i = \frac{1}{2}(1 - \cos\theta_{ir}) \in [0{,}1]$$

$$\hat{\xi}_i = \frac{E_i}{E_{\max}} \in [0{,}1]$$



from R. Poncelet

1. Decomposition of phase space (FKS-inspired)

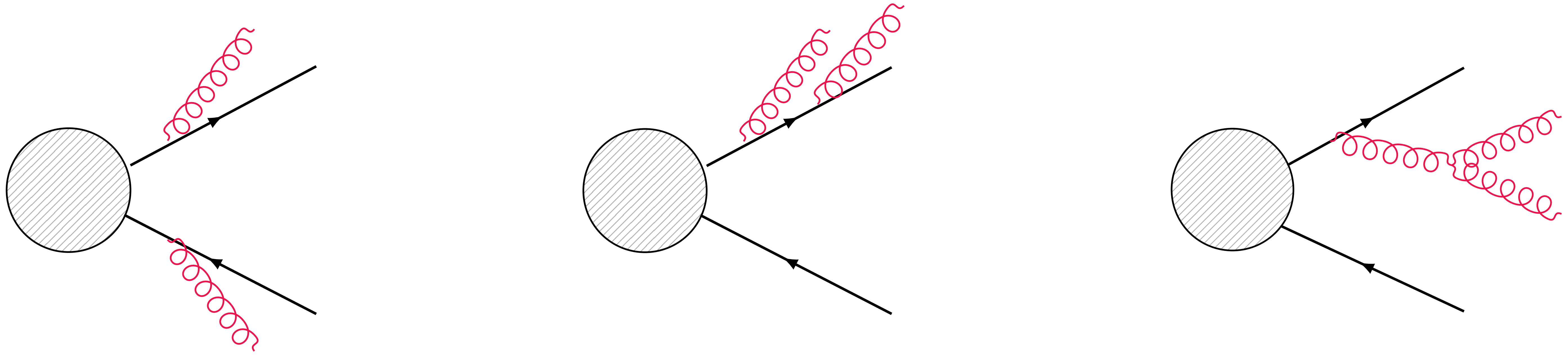
as in STRIPPER [Czakon, Mitov, Poncelet] and Nested Soft-Collinear Subtraction [Caola, Melnikov, Rontsch]

$$1 = \sum_{ij}\left[\sum_{\alpha} w_{ij,\alpha} + \sum_{\alpha\beta} w_{i\alpha;j\beta}\right]$$

triple collinear limit: further splitting since different orderings lead to differs limiting behaviour

Subtraction @ NNLO: anatomy of “complications”

Double real: more involved structure of singular limits. Overlapping of singularities is a more severe problem



double collinear limit triple collinear limit

2. CS-inspired: as CoLoRFulNNLO subtraction **[Bevilacqua, Del Duca, Duhr, Kardos. Somogyi, Sozr, Tramontano, Trocsanyi, Tulipant]**

$$\sigma_{NNLO} = \int d\Phi_{n+2} \left\{ RRF^{n+2} - A_2^{RR}F^n - A_1^{RR}F^{n+1} + A_{12}^{RR}F^n \right\}$$

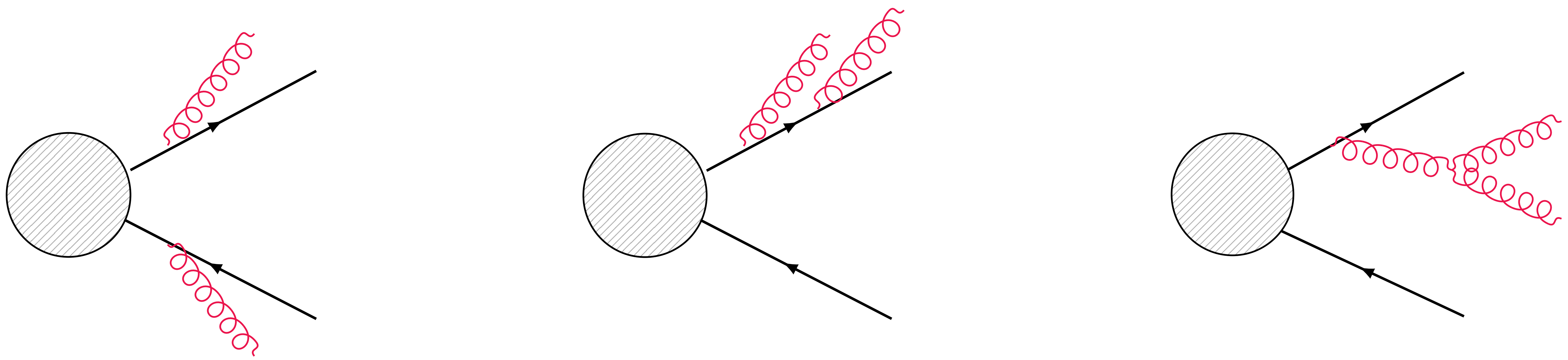
subtract double-unresolved

subtract single-unresolved

remove overlap between A_2^{RR} and and A_1^{RR}

Subtraction @ NNLO: anatomy of "complications"

Double real: more involved structure of singular limits. Overlapping of singularities is a more severe problem



double collinear limit   triple collinear limit

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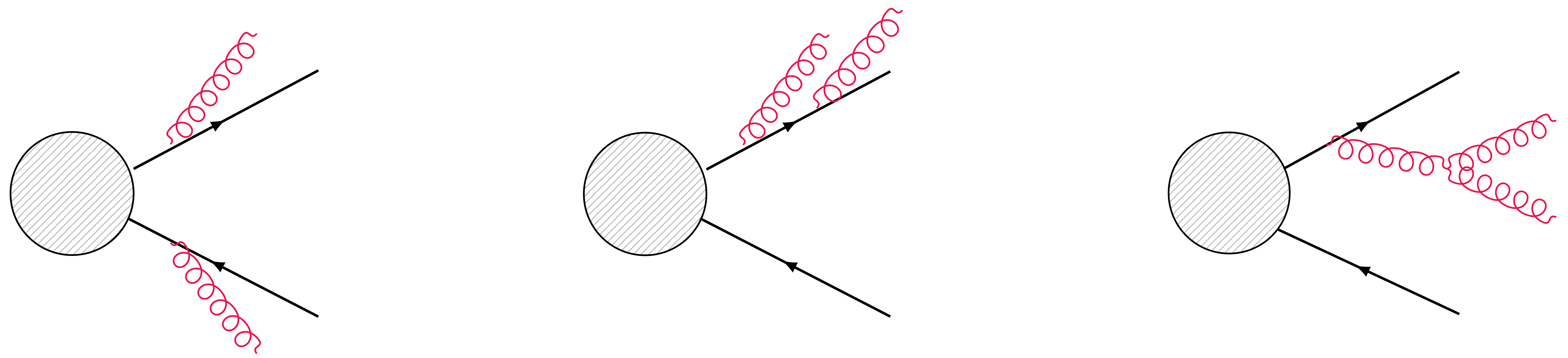
$$+ \int d\Phi_{n+1} \left\{ RVF^{n+1} + \int_1 A_1^{RR}F^{n+1} - A_1^{RV}F^n - \left(\int_1 A_1^{RR} \right)^{A_1} F^n \right\}$$

subtract single-unresolved limit of $\int_1 A_1^{RR}$

Subtraction @ NNLO: anatomy of "complications"

Double real: more involved structure of singular limits. Overlapping of singularities is a more severe problem



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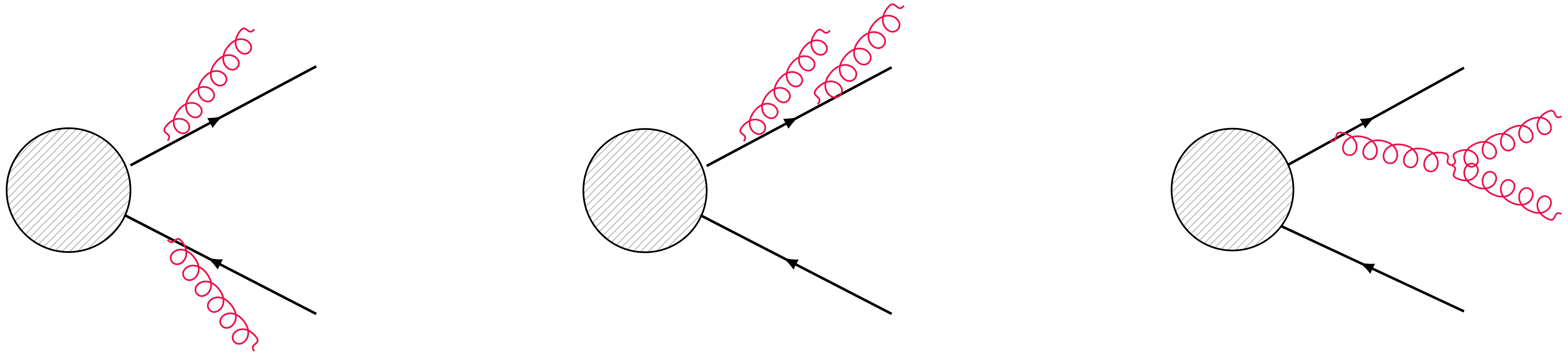
$$+ \int d\Phi_n \left\{ VV + \int_2 \left[A_2^{RR} - A_{12}^{RR} + \right] + \int_1 \left[A_1^{RV} + \left(\int_1 A_1^{RR} \right)^{A_1} \right] \right\} F^n$$

Subtraction @ NNLO: anatomy of "complications"

Double real: more involved structure of singular limits. Overlapping of singularities is a more severe problem



double collinear limit triple collinear limit

2. CS-inspired: as CoLoRFulNNLO subtraction [Bevilacqua, Del Duca, Duhr, Kardos, Somogyi, Sozr, Tramontano, Trocsanyi, Tulipant]

Matching: much more involved; since limits usually do not commute, care must be taken in the choice of ordering

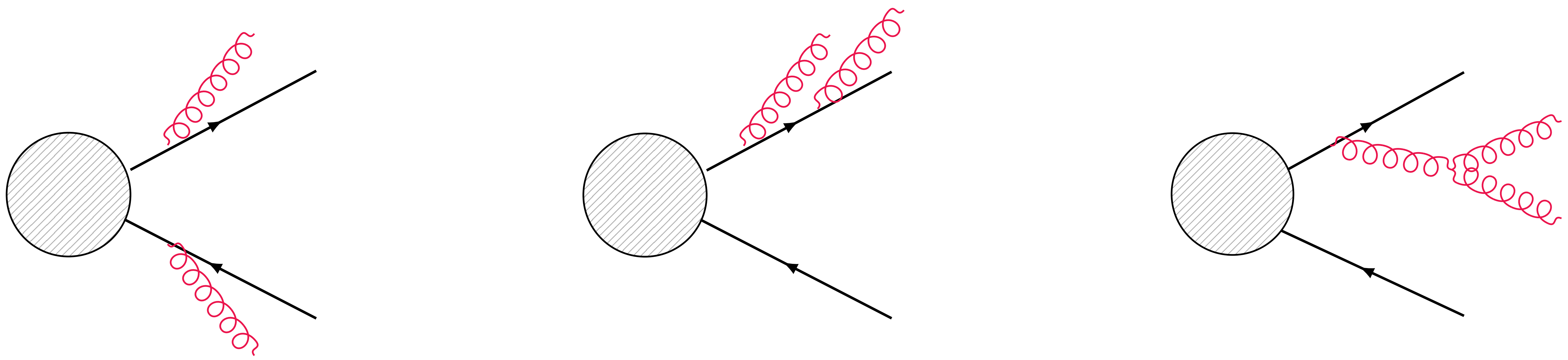
$$A_2 = \sum_{ij} \Big\{ \Big[C_{ij\alpha} + C_{i\alpha,j\beta} + CS_{i\alpha;j} + S_{ij} \Big] - \Big[C_{ij\alpha} \cap CS_{i\alpha;j} + C_{i\alpha;j\beta} \cap CS_{i\alpha;j} + C_{ij\alpha} \cap S_{ij} + CS_{i\alpha;j} \cap S_{ij} + CS_{i\alpha;j\beta} \cap S_{ij} \Big]$$

$$+ \Big[C_{ij\alpha} \cap CS_{i\alpha} \cap S_{j\alpha} + C_{i\alpha;j\beta} \cap CS_{i\alpha;j} \cap S_{ij} \Big] \Big\}$$

from Somogyi talk at Edinburgh 2018 ("Subtracting Infrared Singularities Beyond NLO")

Subtraction @ NNLO: anatomy of "complications"

Double real: more involved structure of singular limits. Overlapping of singularities is a more severe problem



double collinear limit   triple collinear limit

2. CS-inspired: as CoLoRFulNNLO subtraction [Bevilacqua, Del Duca, Duhr, Kardos. Somogyi, Sozr, Tramontano, Trocsanyi, Tulipant]

Extension: requires momentum mappings that respect factorization and delicate structure of cancellations in all limits

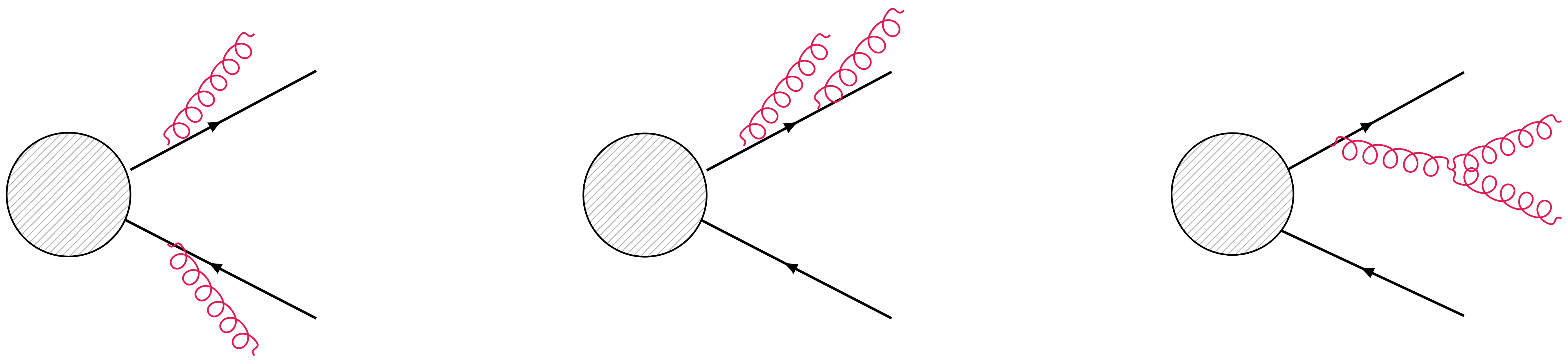
$$\{p\}_{n+1} \to \{\tilde{p}\}_n$$

$$\{p\}_{n+2} \to \{\tilde{p}\}_n$$

Integration: can be tedious and non-trivial

Subtraction @ NNLO: anatomy of "complications"

Double real: **outliers** and **mis-binning** are more severe at NNLO



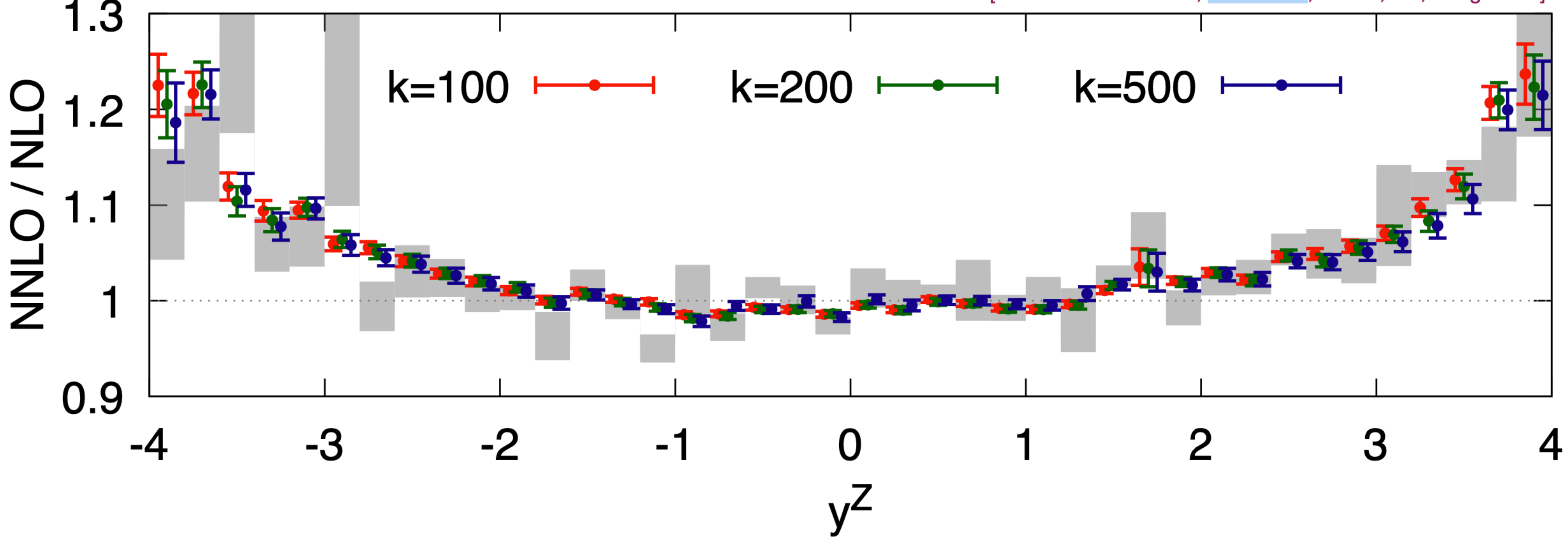
double collinear limit triple collinear limit

Parallelisation is crucial to keep running time manageable

Averaging the results obtained in numerous smaller size samples can lead to large errors because of outliers from mis-binning

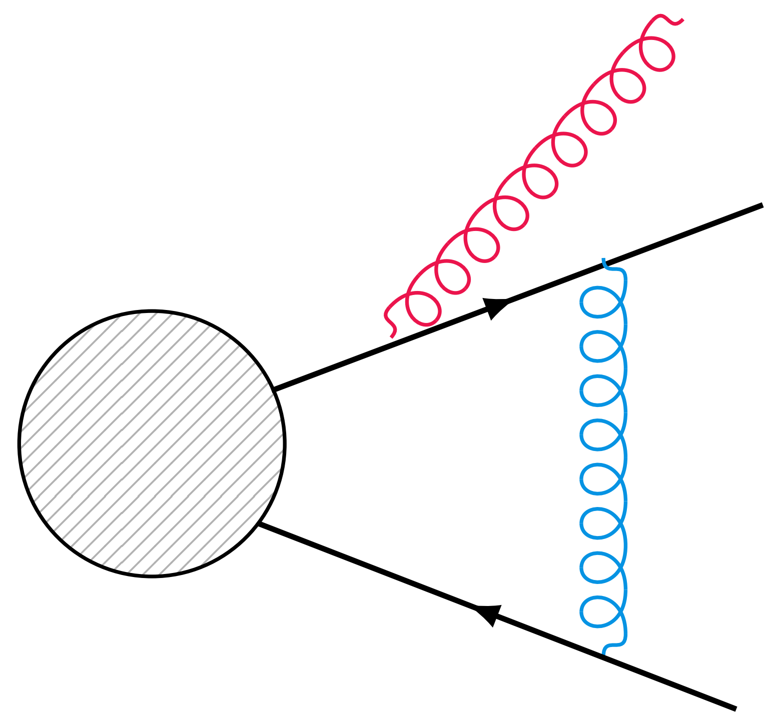
Careful treatment of outliers for obtaining smoother distributions without introducing biases

[Gehrmann-De Ridder, Gehrmann, Glover, AH, Morgan '16]



Subtraction @ NNLO: anatomy of “complications”

Real-virt



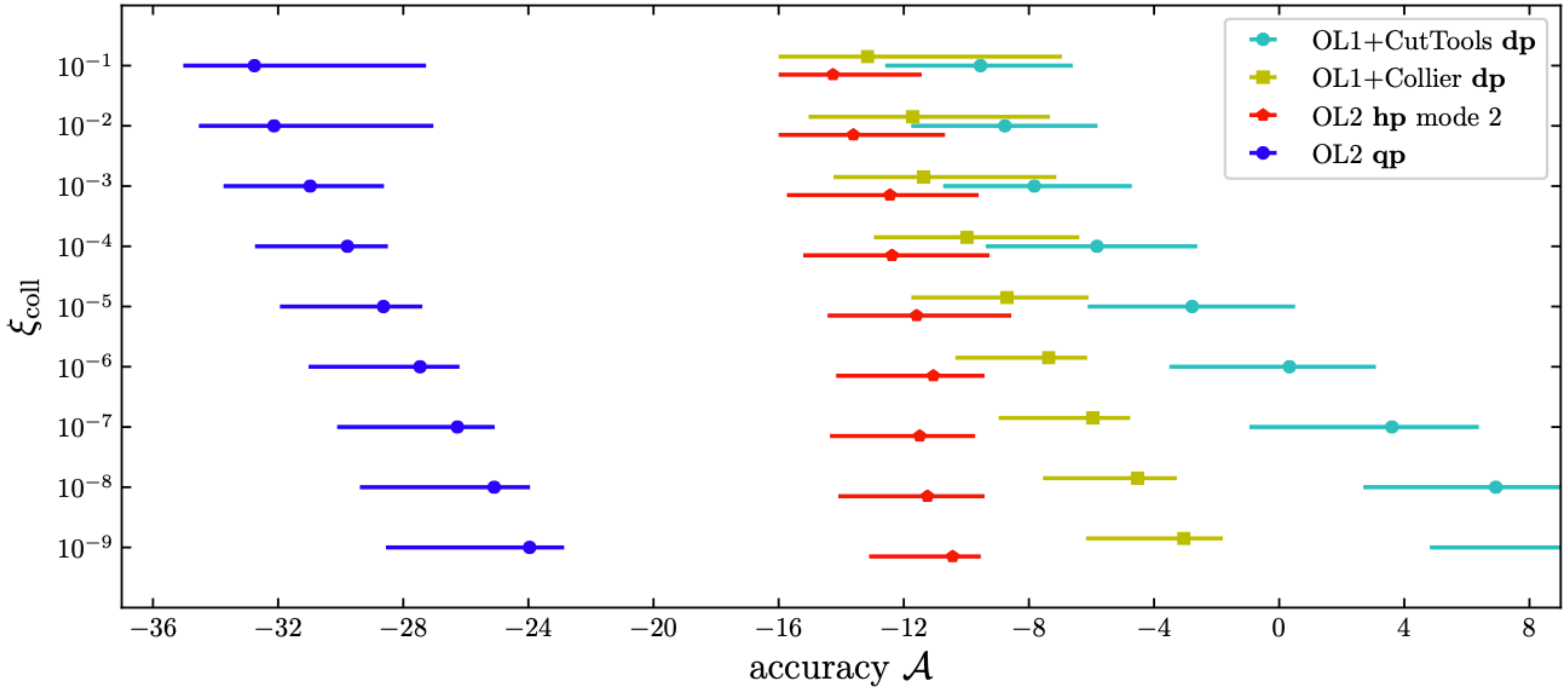
numerical stability is an important issue, especially when probing unresolved regions

Progress in one-loop providers very important

- automated generation of matrix elements for relatively difficult processes (in QCD and in EW)
- stable numerical evaluation suitable for their integration in a NNLO calculation

[Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller '19]

intial-state collinear radiation in gg $\to t\bar{t}g$ at $\mathcal{O}(\alpha_s^4)$



Rescue system

double precision → hybrid precision
$\mathcal{O}(2-10)$ penalty factor in evaluation time

double precision → quad precision
$\mathcal{O}(10-100)$ penalty factor in evaluation time

Subtraction @ NNLO: anatomy of "complications"

Numerical complexity of NNLO calculations: medium/large size HPC clusters required

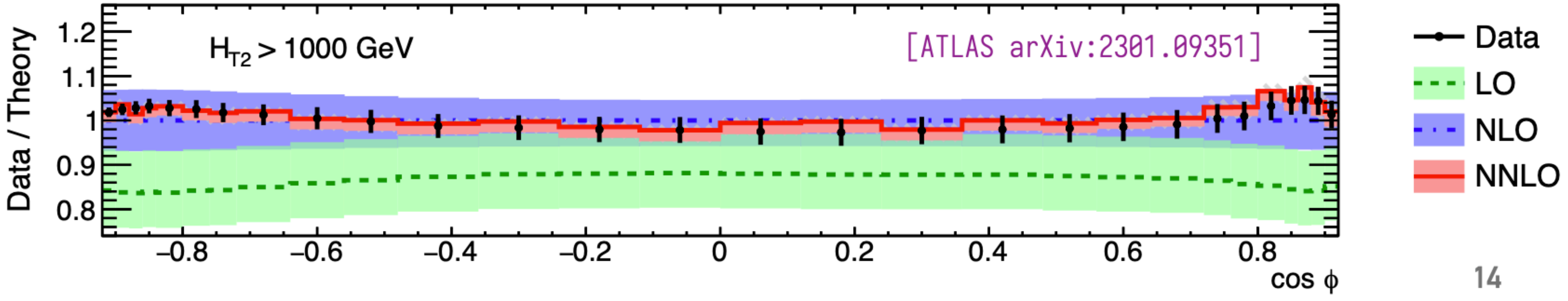
- typical runtime for $2 \to 2$: $\mathcal{O}(100k)$ CPU hours

  $V + j$, di-jet, … → VV:RV:RR ~ 1:20:100

- extreme $2 \to 3$ case: $\mathcal{O}(100M)$ CPU hours

  tri-jet, … → VV:RV:RR ~ 1:100:200

$$\frac{1}{\sigma}\frac{d\Sigma}{d\cos\phi}$$



Different subtraction schemes available on the market with their strengths and limitations but **yet no general frameworks as at NLO** (a lot of activities in this direction)

Remarks

ISSUE: Monte Carlo integration required; how to achieve the cancellation of intermediate singularities while retaining the flexibility of the numerical approach?

- Presentation limited to the “standard approach”: start from real radiation, introduce counterterms, integrate them over radiation phase space, combine with lower-multiplicity contribution

- Alternatively, real and virtual can be integrated simultaneously, for example, using **loop-tree duality relations**

- Integration of counterterms, especially at NNLO, can be highly non-trivial;
  Methods as **reverse unitary** can be exploit to transform phase space integrals into (multi)-loop ones,
  so that multi-loop techniques can be applied to perform this task