Basics of Perturbative QCD

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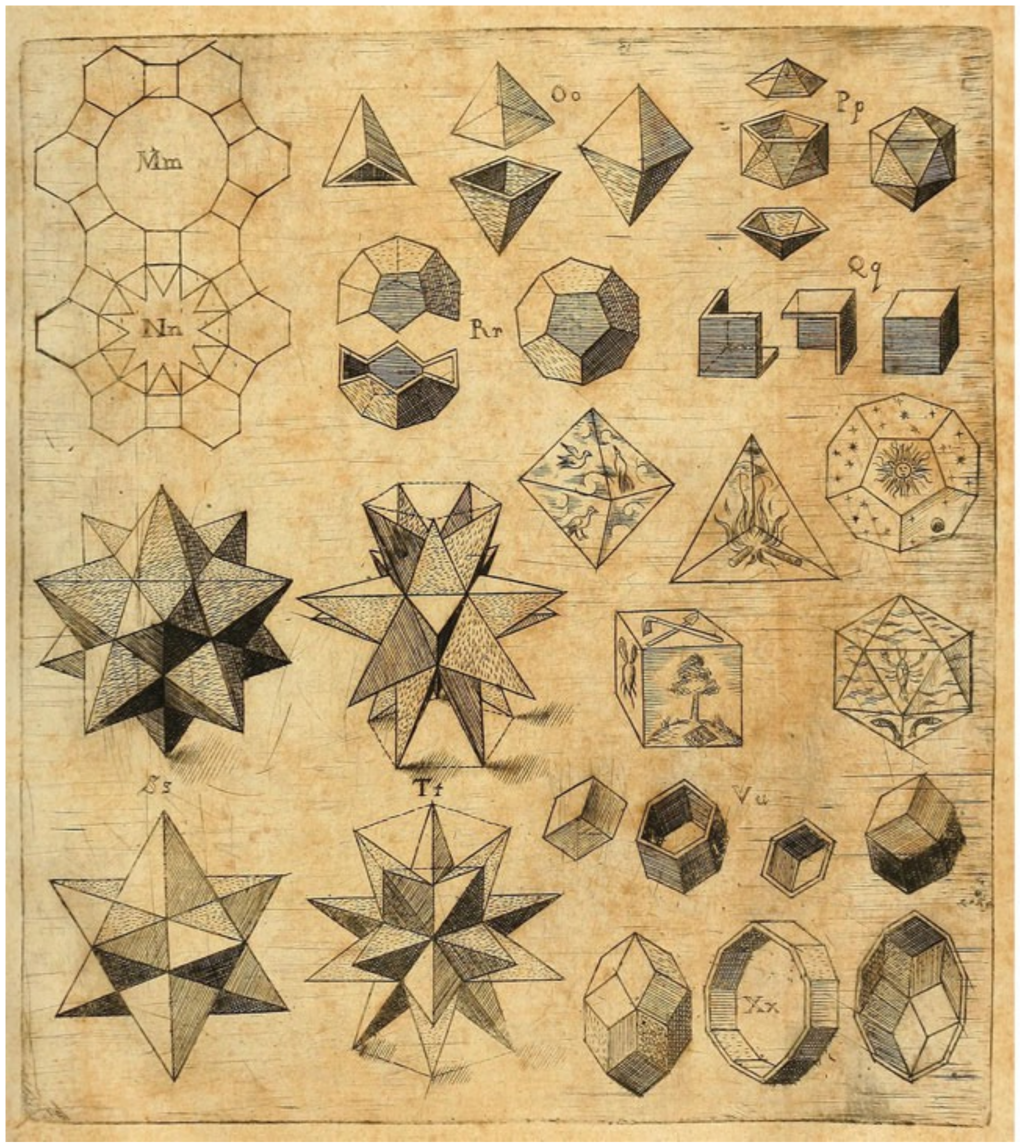
Plan

- Quark Model, Form Factor etc
- Deep Inelastic Scattering
    - Bjorken Scaling
    - Naive Parton Model
- Quantum Chromodynamics (QCD)
- QCD improved Parton Model
    - NLO Coefficient
    - DGLAP evolution
- NNLO and Beyond
- Threshold Resummation

Question:

What is everything made up ?
What are the fundamental building blocks?



Earliest answer by Plato:

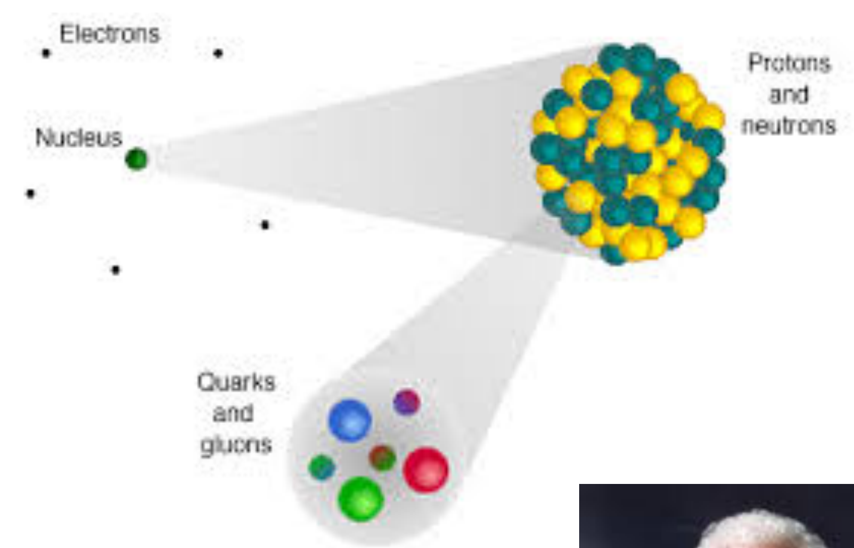
Earth, Air, Fire and Water

Structure of Matter

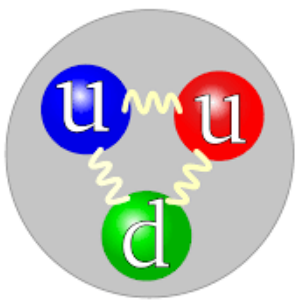
Quarks and Leptons

Atoms

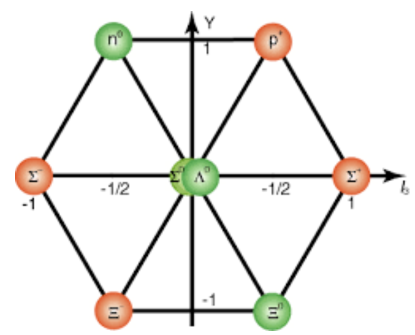
- Graviation
- Electromagnetic
- Weak
- Strong interaction

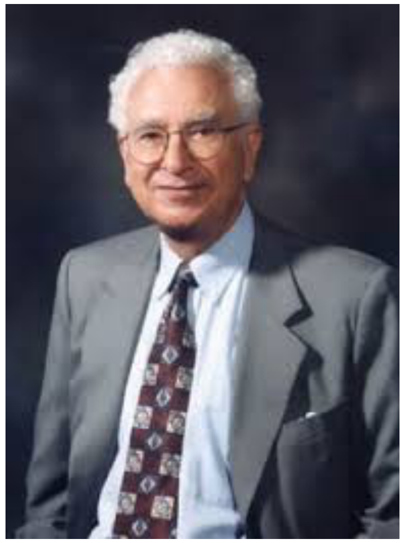


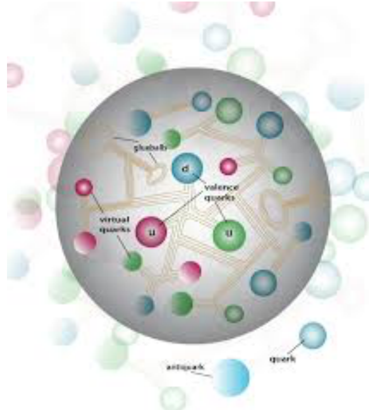
- Strong interaction



Quark Model







Parton Model

QCD

Journey Continues .....

Quark Model

Strongly interacting particles: Hadrons

Mesons and Baryons

Quark Model - Gell-Mann and Zweig

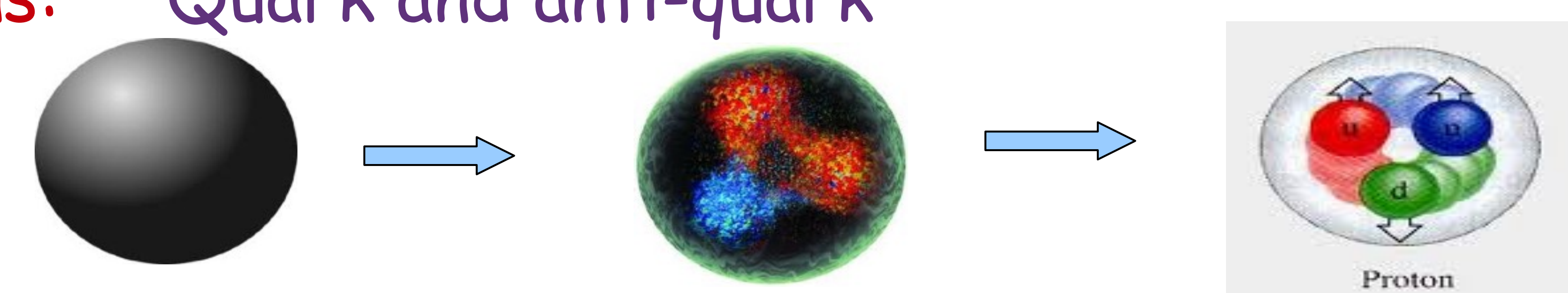
Classification:

Assumes that they are composite objects Made up of point-like spin - 1/2 partilces called

Quarks and Anti-quarks

Baryons: Three quarks

Mesons: Quark and anti-quark



Proton

Quark Model

Gell-Mann and Zweig

Quarks and antiquarks are constituents of hadrons and are spin-1/2 particles carrying fractional charges

Realisation of the model through Symmetry group: Flavour-SUf(3)

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

-group of special unitary matrices

$$U(\vec{\alpha}) \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} u' \\ d' \\ s' \end{pmatrix}, \quad U(\vec{\alpha}) = \exp\left(i\vec{\alpha}\cdot\vec{t}\right) \in SU_f(3)$$

$$U^\dagger U = 1, \quad detU = 1$$

$(t^1, t^2, \cdots t^8)$ - generators

$[t^a, t^b] = if^{abc}t^c, \quad f^{abc}$ - structure constants

- Mesons: bound states of a quark and an antiquark
- Baryons: bound states of three quarks

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

$J^P = 0^-$: $(\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta)$

$J^P = 1^-$: $(\rho^\pm, \rho^0, K^{*\pm}, K^{*0}, \bar{K}^{*0}, \omega)$,

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{10} + \mathbf{8} + \mathbf{8} + \mathbf{1}$$

Baryon octet, $J^P = 1/2^+$: $(p, n, \Sigma^\pm, \Sigma^0, \Xi^-, \Xi^0, \Lambda)$.

Baryon decouplet, $J^P = 3/2^+$: $(\Delta^{++}, \Delta^+, \Delta^0, \Delta^-, \Sigma^\pm, \Sigma^0, \Xi^-, \Xi^0, \Omega^-)$

Color Quantum Number

Problem with Fermi-Dirac Statisics

Δ^{++} $u\uparrow u\uparrow u\uparrow$

Quark Model predicts fully symmetric wave function

Spacial part is symmetric

Flavor and Spin part is symmetric (no anti-symmetric combination)

Need for New Quantum number: Color Quantum number

$$q_a \to q_{a,i} \quad i = 1, 2, 3$$

Han and Nambu, Greenberg and Gell-Mann

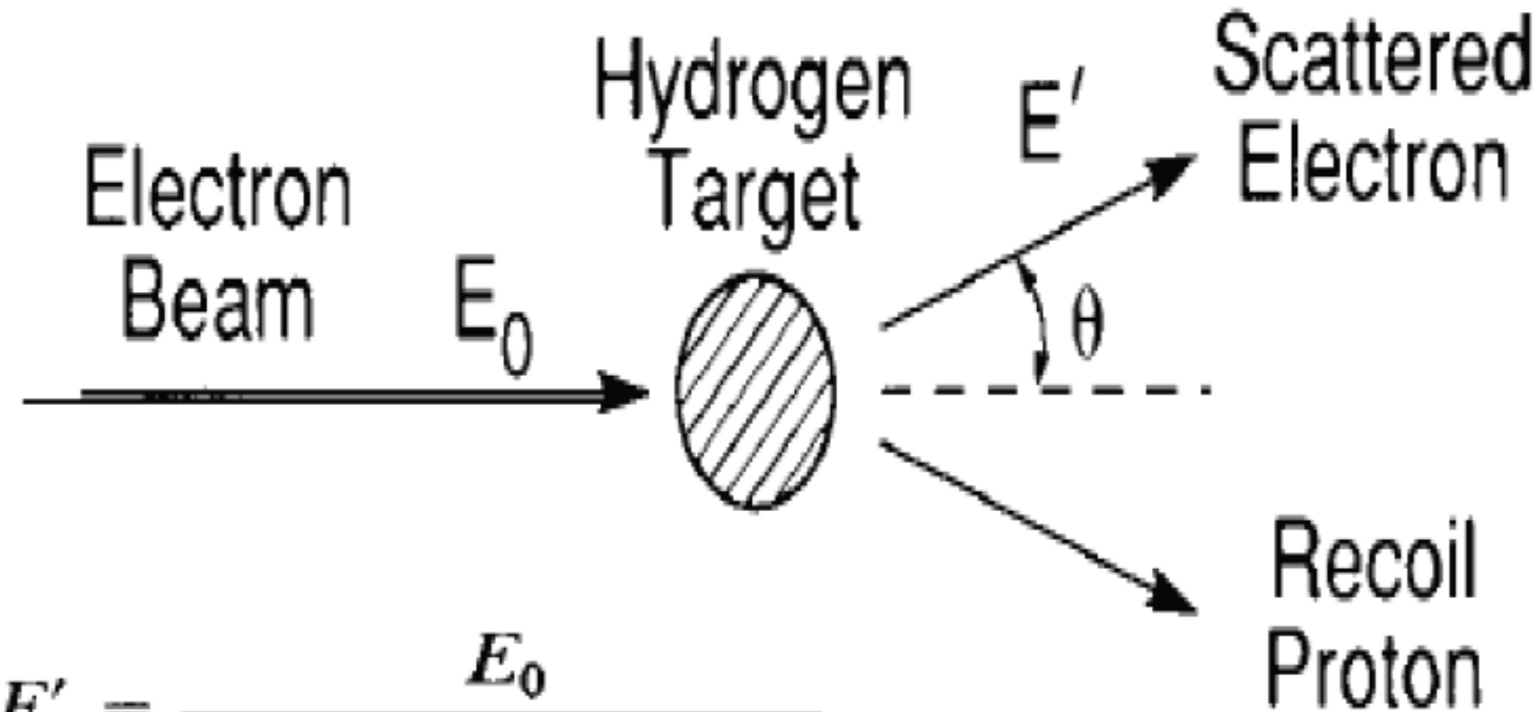
a = u,d and s each has (red,green,blue)

Use this color quantum number to construct anti-symmetric combination to resolve problem with Fermi-Dirac statistics

$$\epsilon_{ijk} u^i u^j u^k$$

Birth of **Quantum Chromodynamics**

Electron-proton elastic scattering:



$$E' = \frac{E_0}{1 + \frac{2E_0}{M}\sin^2\theta/2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \cdot \frac{E'}{E}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right]$$

The functions G_E and G_M take account of the size of the proton; they are called ***form factors***

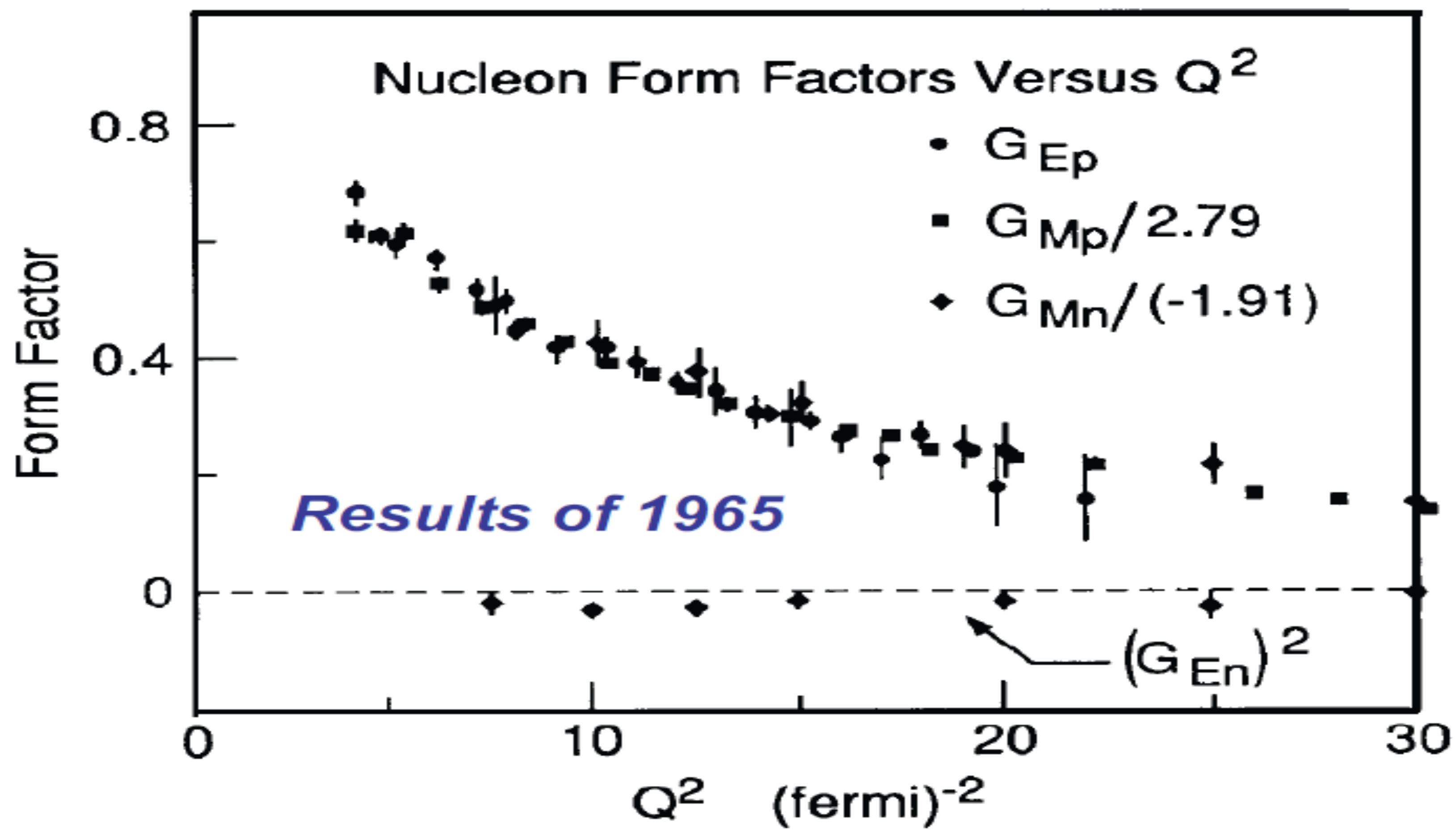


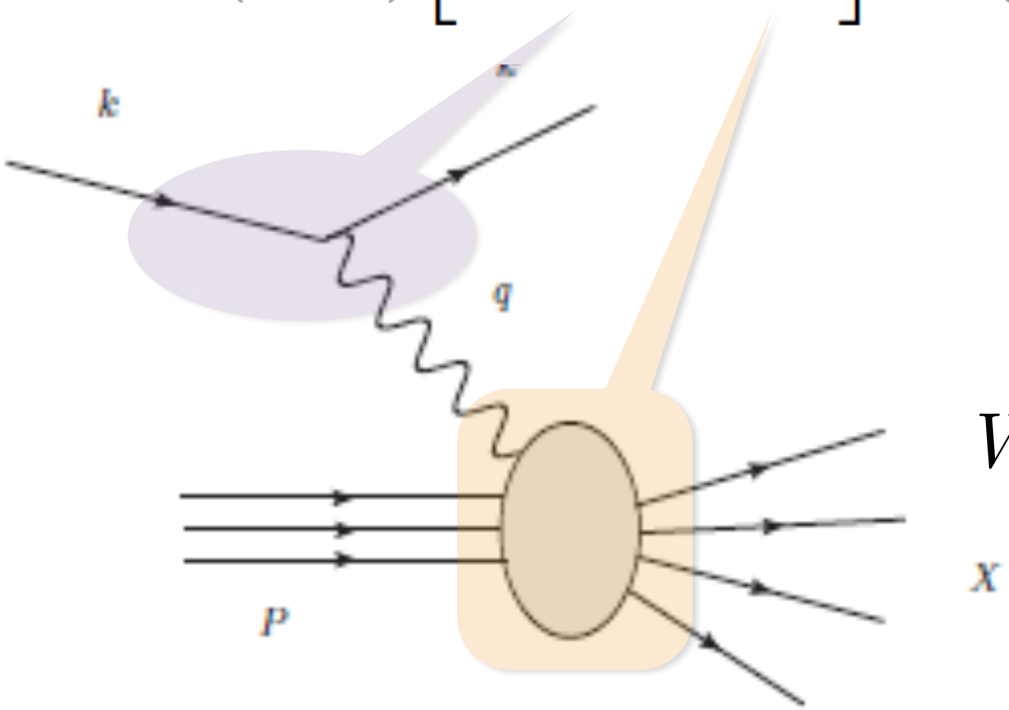
Fig. 23. Summary of results on nuclear form factors presented by the Stanford group at the 1965 "International Symposium on Electron and Photon Interactions at High Energies". (A momentum transfer of 1 GeV^2 is equivalent to 26 Fermis^{-2}.)

$$G_{Ep}(Q^2) \cong \left(\frac{1}{1 + \frac{Q^2}{0.71\ \mathrm{GeV}^2}} \right)^2 \text{up to } Q^2 \sim 10\ \mathrm{GeV}^2$$

Structure Functions from DIS

Inelastic Scattering Factorises

$$d\sigma = \frac{1}{4(k\cdot P)}\left[\frac{4\pi e^4}{q^4}L_{\mu\nu}W^{\mu\nu}\right]\frac{d^3k'}{2E'(2\pi)^3}$$



Leptonic Tensor

$$L_{\mu\nu} = 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \frac{Q^2}{2}g_{\mu\nu}\right]$$

Hadronic Tensor

$$W^{\mu\nu} = \left(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\right)W_1 + \left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)W_2$$

$W_i(\nu, Q^2) \quad i = 1, 2$ Structure Function

$$x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xys$$

NOT CALCULABLE

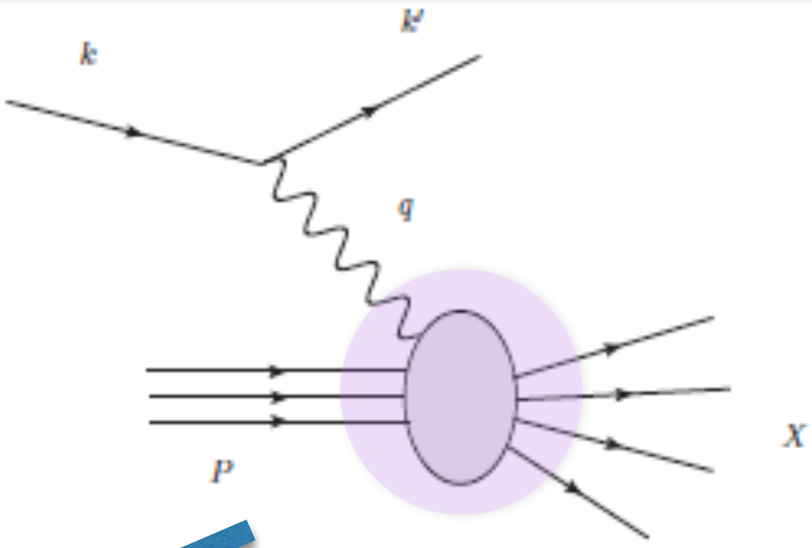
Inclusive Cross section

$$\frac{d\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\frac{\theta}{2}}\left(W_2(\nu, Q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, Q^2)\sin^2\frac{\theta}{2}\right)$$

$$m_p W_1(\nu, Q^2) \to F_1(x) \qquad \nu W_2(\nu, Q^2) \to F_2(x)$$

Bjorken Scaling

Deep Inelastic Scattering



Hadronic Tensor

$$W^{\mu\nu}(P,q) = \int d^4\xi \; e^{iq\cdot\xi} < P|J^\mu(\xi)J^\nu(0)|P >$$
$$= \Big(-g^{\nu\mu} + \frac{q^\nu q^\mu}{q^2}\Big)W_1 + \Big(P^\nu - \frac{P\cdot q}{q^2}q^\nu\Big)\Big(P^\mu - \frac{P\cdot q}{q^2}q^\mu\Big)W_2$$

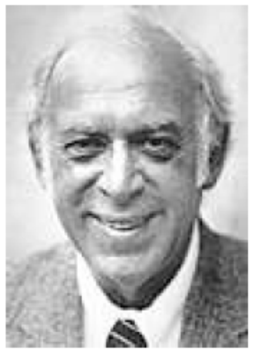
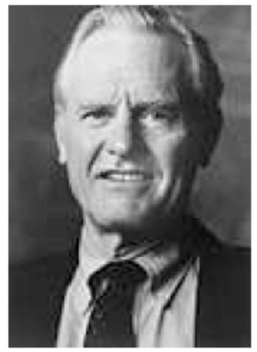
Bjorken Limit: $-q^2 \to \infty, P\cdot q \to \infty$ *with* $x = \dfrac{-q^2}{2P.q}$ *fixed*

$$W_1(P,q) = F_1(x),$$
$$P.q\; W_2(P,q) = F_2(x)$$

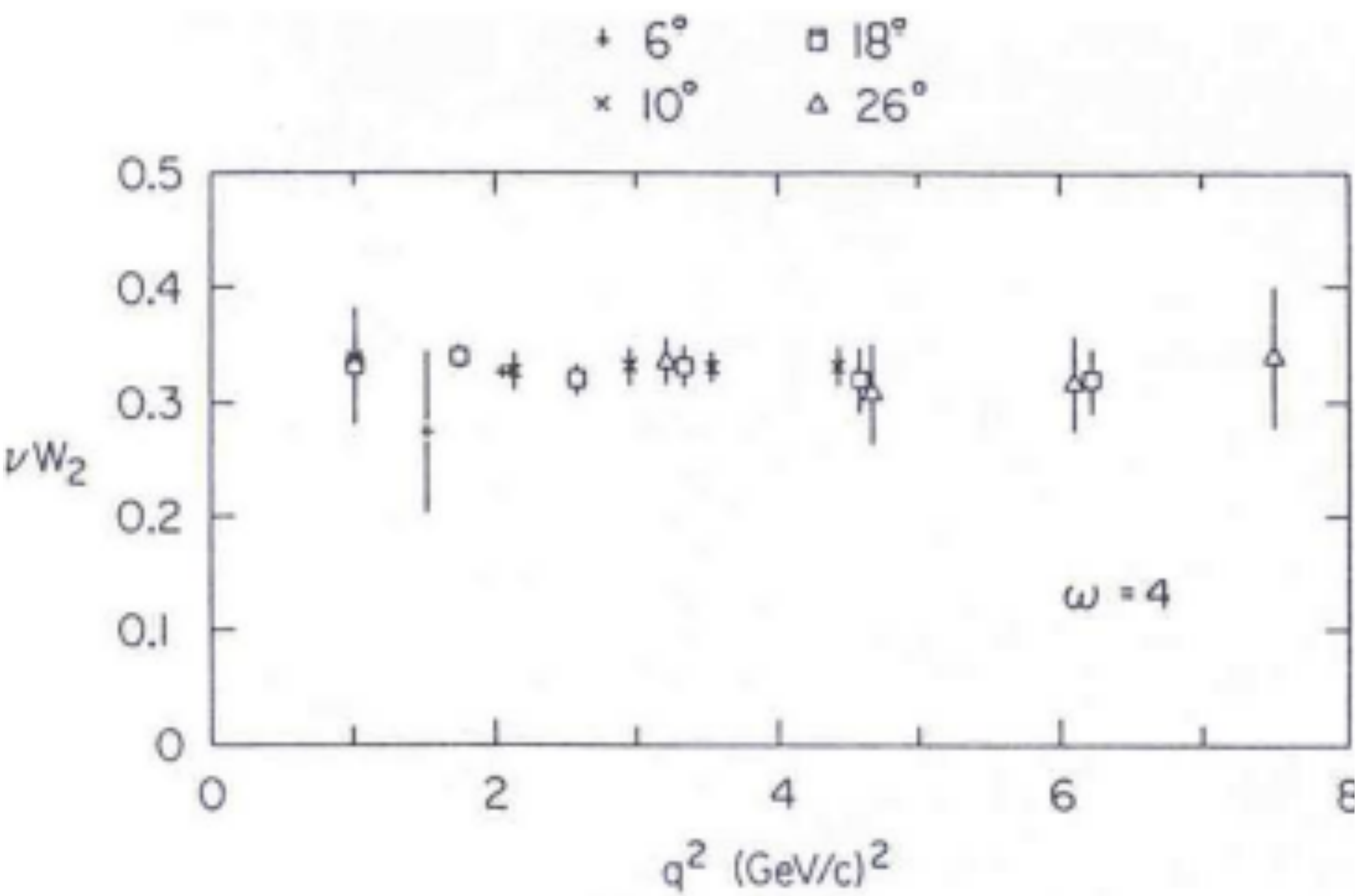
Bjorken Scaling

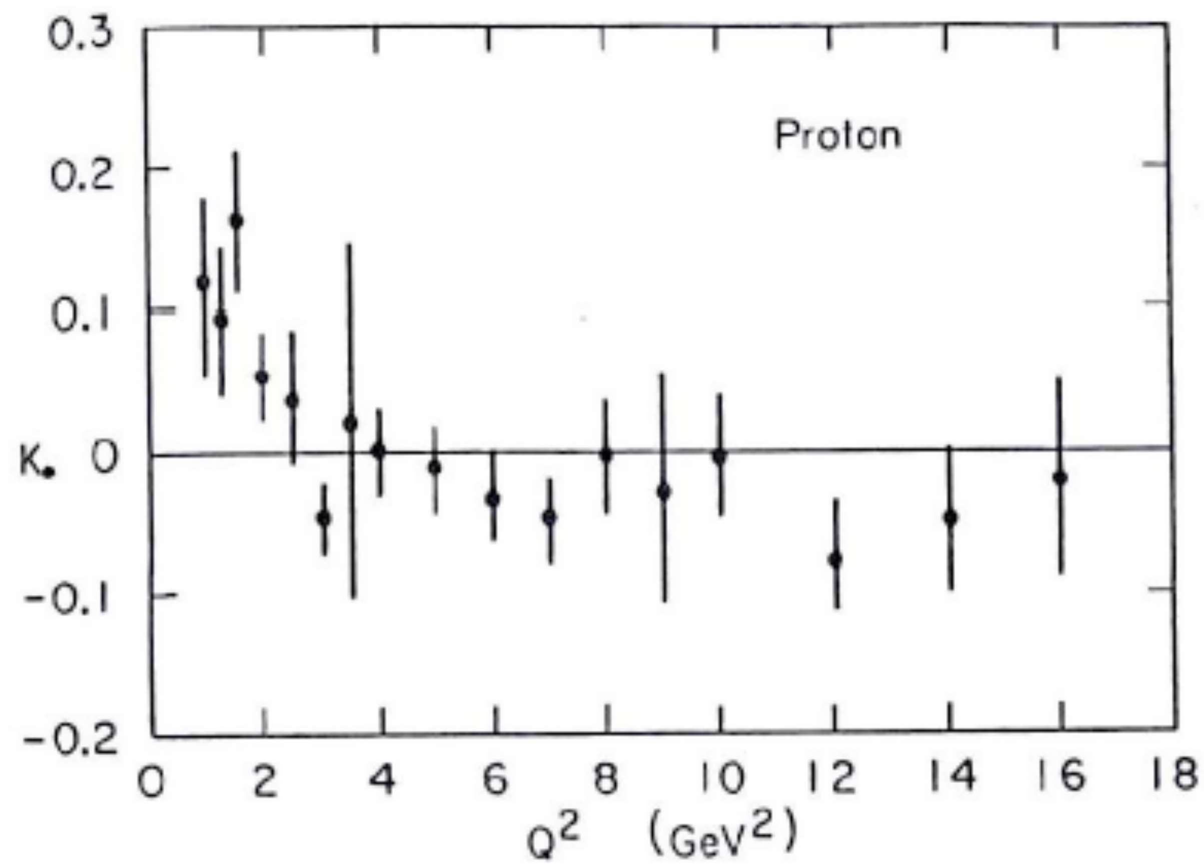
Birth of PARTON MODEL

Deep Inelastic Scattering

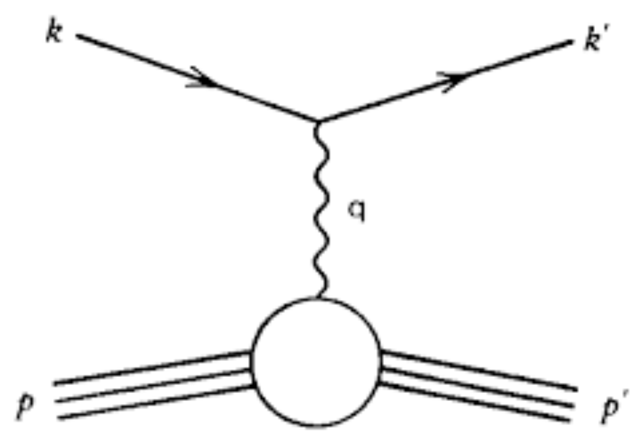
$Q^2 \approx 3M_N^2$ J. Friedman *1930 H. Kendall (1926-1999) R. Taylor *1929 (1968/69)

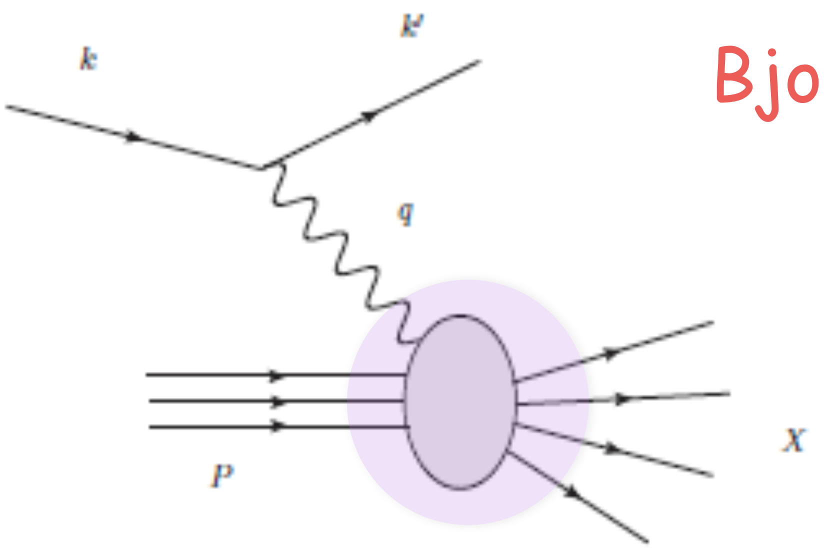




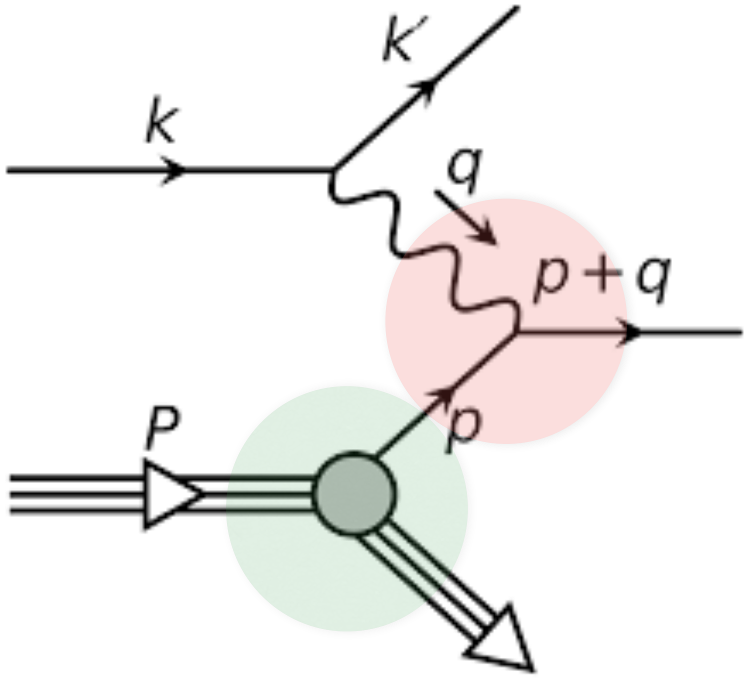
Parton Model

Elastic Scattering





Bjorken Scaling:



Deep Inelastic Scattering

PARTON MODEL PICTURE

Parton Model

Note that:

- Hadrons are extended objects.

At high energies (infinite momentum frame):

- Hadron is Lorentz contracted in the direction of collision
- Interaction between constituents is time dilated

Model assumes:

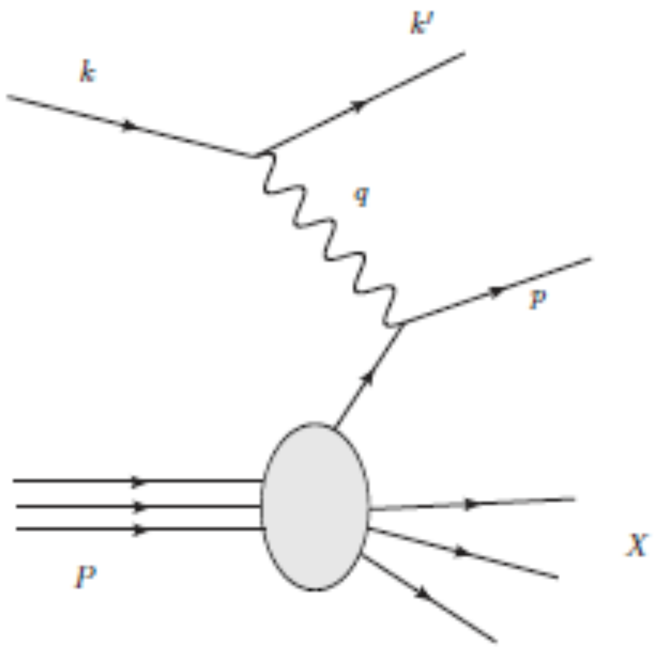
- At high energies they look like a collection of point like particles called partons held together by mutual interactions.
- Hadrons can be thought of one of the virtual states of these partons

Time dilation:

- Electron/photon interaction with partons takes place in a time scale shorter compared to time scales of virtual parton states.

Result: Inelastic scattering of electron proton can be thought of as a incoherent sum of elastic scattering of electron and a parton.

Naive Parton Model



$$d\sigma^{DIS}(P,q) = \sum_i \int_x^1 dz\, f_i(z)\, d\hat{\sigma}_i(zP,q)$$

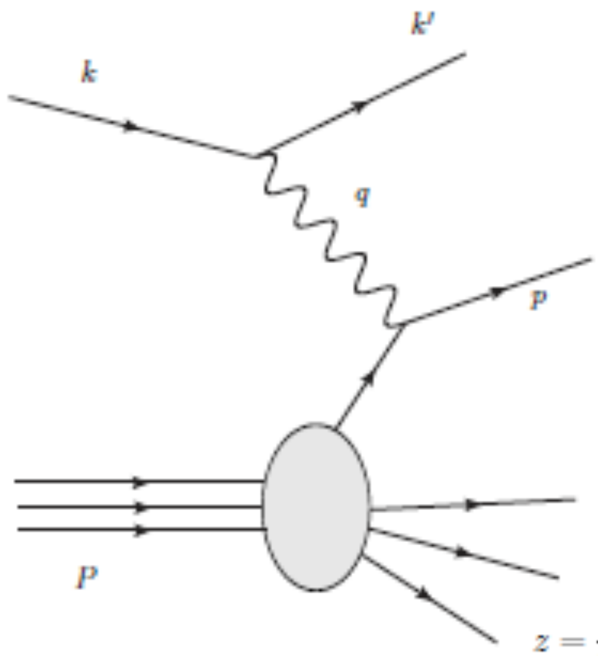
- Elastic scattering cross section with i-th parton
- Does not depend on the details of the target proton - Target Independent

$f_i(z)$ Parton Distribution Function (PDF)

- Probability of finding i-th parton with momentum fraction z of proton

- Does not depend on the future course of action of the i-th parton - Process Independent

Parton Model



Parton Model - Master Formula

$$W_{\mu\nu}(P,q) = \sum_i \int_0^1 \frac{dy}{y} f_i(y) \hat{W}^{(i)}_{\mu\nu}(yP,q)\,,$$

$$x = \frac{Q^2}{2p.q}; \quad y = \frac{p.q}{p.k}; \quad Q^2 = xys$$

Dimension-less

$$m_P W_1(\nu, Q^2) = F_1(x, Q^2),$$
$$\nu W_2(\nu, Q^2) = F_2(x, Q^2)$$

$$F_2(x,Q^2) = \sum_i \int_x^1 \frac{dy}{y} f_i(y) \hat{F}_2(x/y, Q^2)\,,$$

Parton level Cross sections

$$\hat{F}_1(x) = \frac{1}{2} e_q^2 \delta(x - \xi)\,,$$
$$\hat{F}_2(x) - 2x\hat{F}_1(x) = 0\,.$$

Bjorken scaling

$$2xF_1(x) = F_2(x) = \sum_i e_i^2\ x\ f_i(x)$$

Momentum sum rule

$$2xF_1(x) = F_2(x) = \sum_i Q_i^2 \; x \; f_i(x)$$

Measurements for proton and neutron

$$\int_0^1 F_2^p(x)dx = \frac{4}{9}f_u + \frac{1}{9}f_d = 0.18$$

$$\int_0^1 F_2^n(x)dx = \frac{4}{9}f_d + \frac{1}{9}f_u = 0.12$$

SU(2) symmetry

where $f_q = \int_0^1 dx \; x \; f_q(x)$

Contribution to hadron momentum

$f_u = 0.36$ $\quad f_d = 0.18$

Only about 50% from quarks!

We now know that

Gluons also contribute significantly to momentum

what are these gluons?

Quantum Chromodynamics

Han and Nambu, Greenberg and Gell-Mann

Solution to Δ^{++} $u\uparrow u\uparrow u\uparrow$ came from

Symmetry group: Color-SUc(3)

$$U(\vec{\beta}) = \exp\left(i\vec{\beta}\cdot\vec{T}\right) \in SU_c(3)$$

$$U(\vec{\beta})\begin{pmatrix} q_{i,1} \\ q_{i,2} \\ q_{i,3}\end{pmatrix} = \begin{pmatrix} q'_{i,1} \\ q'_{i,2} \\ q'_{i,3}\end{pmatrix}$$

$SU(N)$

$$[T^a, T^b] = i f^{abc} T^c \qquad i = 1,\cdots,N \qquad a = 1,\cdots,N^2-1$$

Casimirs

$$(T^a T^a)_{ij} = C_F \delta_{ij}$$

$$f^{abc} f^{a'bc} = C_A \delta^{a'a}$$

$$C_F = \frac{N^2-1}{2N}, \quad C_A = N$$

Gauge the symmetry group SUc(3)

Gauge Theory of strong interaction

QUANTUM CHROMODYNAMICS

Matter Field:s ψ_i , $\overline{\psi}_i$ - Fundamental representation of SUc(3)

Gauge Fields A^a_μ - Adjoint representation

Gauge symmetry

QCD Lagrangian

$$\mathcal{L}_{QCD} = \overline{\psi}_j \left(i\not{D}_{jk} - m\delta_{jk} \right) \Psi_k - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \text{G.F}$$

where $D_\mu \psi(x) = \left(\partial_\mu - i g_s T^a A^a_\mu \right) \psi(x)$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_s [A_\mu, A_\nu]$$

$$A_\mu = T^a A^a_\mu$$

Invariant under

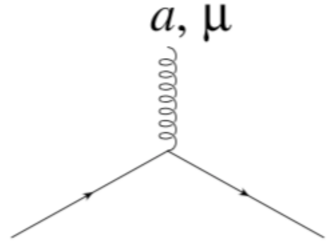
$$A_\mu \to U(\beta) \left(A_\mu - \frac{i}{g} \partial_\mu \right) U^\dagger(\beta)$$

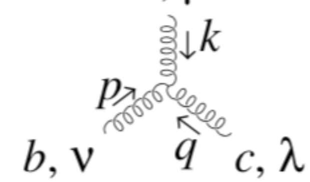
$$D_\mu \psi \to U(\beta) D_\mu \psi$$

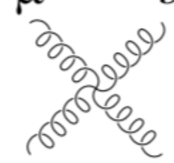
$$U(\vec{\beta}) \begin{pmatrix} q_{i,1} \\ q_{i,2} \\ q_{i,3} \end{pmatrix} = \begin{pmatrix} q'_{i,1} \\ q'_{i,2} \\ q'_{i,3} \end{pmatrix}$$

K.E + Interaction part

$$\mathcal{L}_{QCD} = \mathcal{L}_{K.E} + g_s A^a_\mu \overline{\psi} \gamma^\mu T^a \psi - g_s f^{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} - g_s^2 f^{eab} f^{ecd} A^a_\mu A^b_\nu A^{\mu c} A^{\nu d}$$





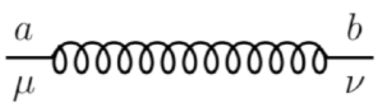


where

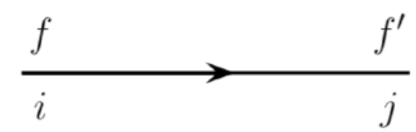
$$\mathcal{L}_{K.E} = -\frac{1}{2} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{c}^a)(\partial^\mu c^a) + \sum_f \overline{\Psi}_{if} (i\not{\partial} - m_f) \Psi^{if}$$

Feynman rules

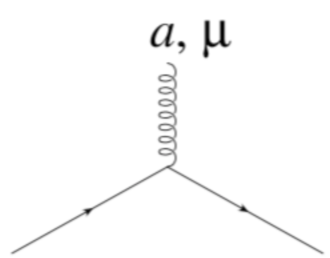
QCD Feynman Rules



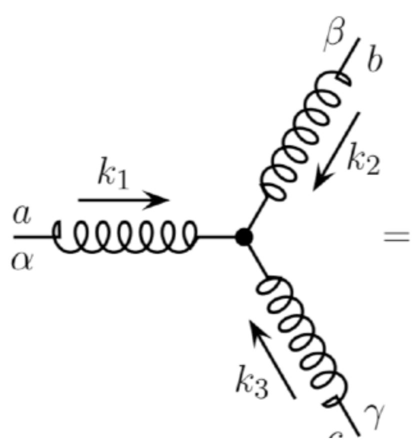
$$= \frac{-i\delta^{ab}}{k^2 + i0}\left(g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right)$$



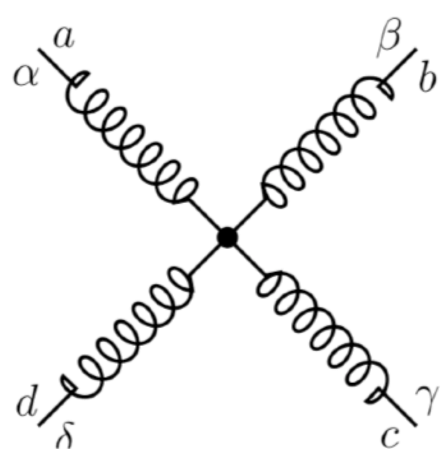
$$= \frac{i\delta^i_j \delta^f_{f'}}{\not{p} - m_f + i0}\,.$$



$$= \qquad ig\gamma^\mu T^a$$



$$= -gf^{abc}\left[g^{\alpha\beta}(k_1 - k_2)^\gamma + g^{\beta\gamma}(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\beta\right]$$



$$= -ig^2\begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}$$

Ghost fields …

Gauge fixing:

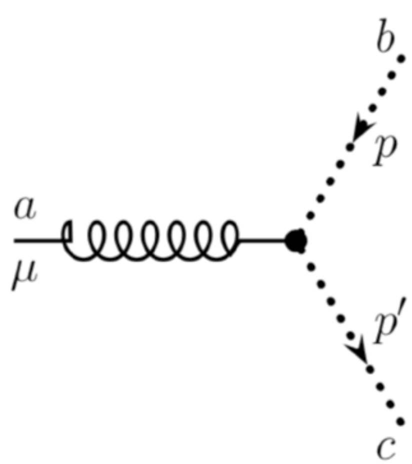
Gauge Fields contain two physical and two unphysical degrees of freedom

Ghost fields propagate as well as interact with gauge bosons but do not show up as physical particles

Ghost terms (Lorenz Gauge: $\partial_\mu A^\mu = 0$):

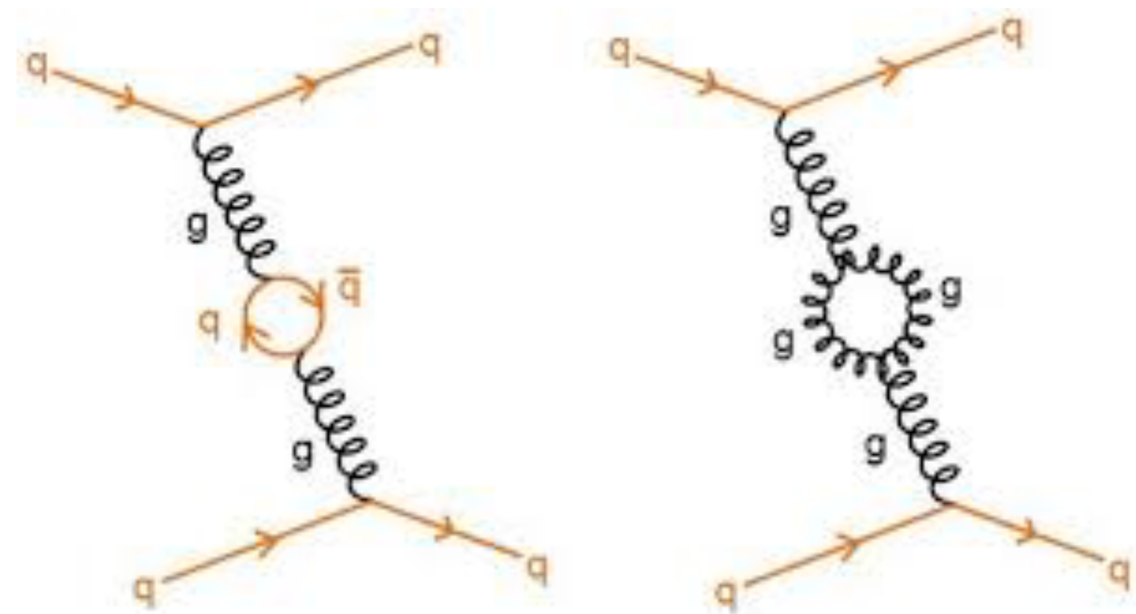
$$\mathcal{L}_{gh} = \partial_\mu \overline{c}^a \partial^\mu c^a - g_s f^{abc} (\partial_\mu \overline{c}^a) A^{\mu b} c^c$$

$$= \frac{i\delta^{ab}}{k^2 + i0} .$$



$$= +g f^{abc} p'^{\mu}$$

Renormalisation

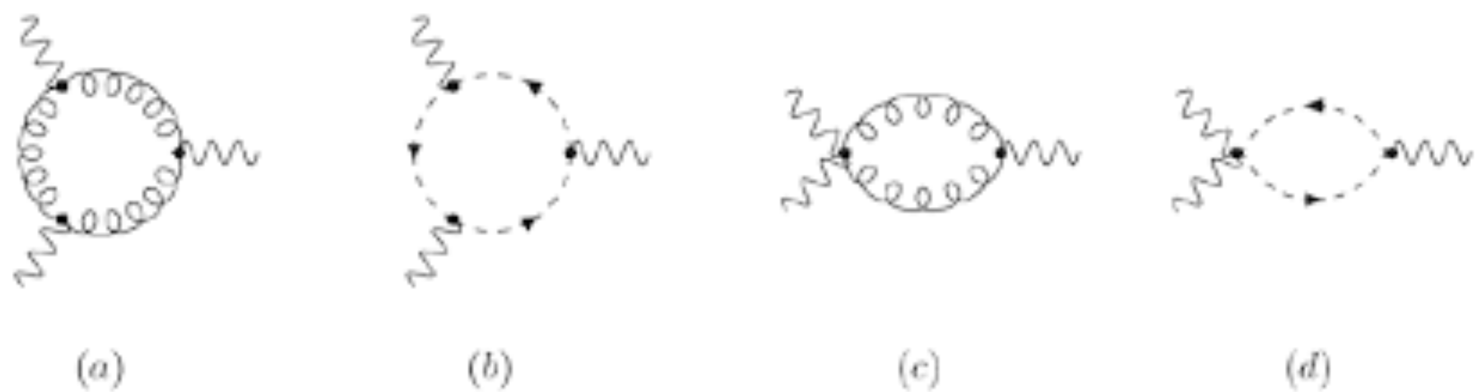


Quantum corrections

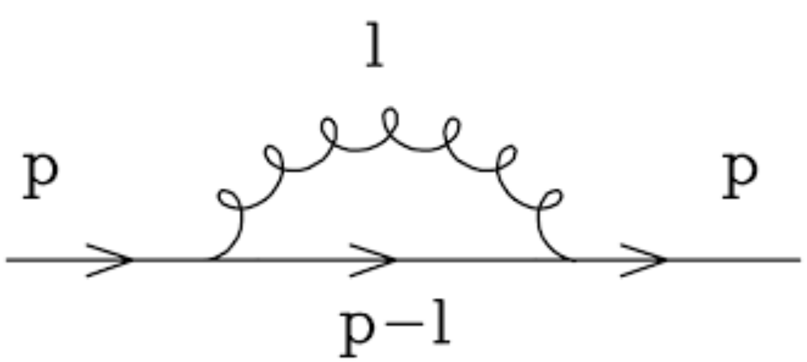
Perturbation theory to compute Green's functions

Vacuum polarization

Vertex Function



Selft energy



$$\int \frac{d^4 l}{(4\pi)^4} \frac{f(l,p)}{(l^2+i\epsilon)((p-l)^2+i\epsilon)}$$

diverge when $l^\mu \to \pm\infty$

Renormalisation

We use Perturbation theory to compute Green's functions

Coupling constant serves as an expansion parameter

Beyond leading order in coupling constant, they often diverge

Large momentum regions in Feynman Loop integrals

At short distances, local interactions give UV divergences

Practical resolution: Renormalisation

1. Regularise the theory so that we can proceed

2. Rescale the fields and parameters with ``would be infinite" constants (Zi) in such a way that the Green's function computed in terms of recalled fields and parameters always give finite result

Counter Terms …

Regularisation: Dimension regularization : $d = 4 + \varepsilon$

d - space time dimension

$$\mathcal{L}_{QCD}\left(\psi, \overline{\psi}, A_\mu, c, \overline{c}, g_s, m\right) \rightarrow \mathcal{L}_{QCD}\left(\psi, \overline{\psi}, A_\mu, c, \overline{c}, g_s, m, \epsilon, \mu\right)$$

g_s dimensionless in d-dimensions

Rescaling of fields and parameters:

$$\psi = Z_\psi^{\frac{1}{2}}(\mu_R)\psi_R(\mu_R)$$

$$A_\mu = Z_A^{\frac{1}{2}}(\mu_R)A_{\mu R}(\mu_R)$$

μ Regularisation scale

μ_R Renormalisation scale

• • • • •

$$g_s = Z_g(\mu_R)\left(\frac{\mu}{\mu_R}\right)^{\frac{\varepsilon}{2}} S_\varepsilon^{-\frac{1}{2}} g_{sR}(\mu_R)$$

$$\mathcal{L}_{QCD}\left(\psi, \cdots, g_s, m, \epsilon, \mu\right) = \mathcal{L}_{QCD,R}\left(\psi_R, \cdots, g_{sR}, m_R, \epsilon, \mu_R\right)$$

$$+\mathcal{L}_{CT}\left(\{Z_i(\mu_R)\}, \psi_R, \cdots, g_{sR}, \epsilon, \mu_R\right)$$

Choose Zi in such a way all the Green's functions in terms of renormalized quantities give finite result to all orders in perturbation theory

Renormalisation Group:

Renoramliation group invariance:

$$\mathcal{L}_{QCD}\left(\psi,\cdots,g_s,m,\epsilon,\mu\right)=\mathcal{L}_{QCD,R}\left(\psi_R,\cdots,g_{sR},m_R,\epsilon,\mu_R\right)$$
$$+\mathcal{L}_{CT}\left(\{Z_i(\mu_R)\},\psi_R,\cdots,g_{sR},\epsilon,\mu_R\right)$$

Since LHS does not depends on μ_R

Green's functions computed from RHS will also not depend on μ_R, when it is computed to all orders in coupling constant

This implies $$\mu_R^2\frac{d}{d\mu_R^2}<\Omega|T\{\psi\cdots A_\mu\cdots\overline{\psi}\cdot\cdot\}|\Omega>=0$$

This leads to renormalisation group equations for

$$\mu_R^2\frac{d}{d\mu_R^2}\log\left(<\Omega|T\{\psi_R\cdots A_{\mu,R}\cdots\overline{\psi}_R\cdot\cdot\}|\Omega>\right)=\Gamma(\mu_R)$$

in the limit $\varepsilon\to 0$

One Loop result

Renormalisation group equation:

$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = -a_s(\mu_R^2)\mu_R^2 \frac{d}{d\mu_R^2} \log(Z_a(\mu_R^2)) = \beta(a_s)$$

Beta function (exact)

$$\beta(a_s) = a_s \frac{\varepsilon}{2} \frac{1}{1 + a_s \frac{d}{da_s} \log Z_a}$$

Perturbative result to one-loop:

$$Z_a = 1 + a_s(\mu_R^2)\frac{1}{\varepsilon}\beta_0 + \mathcal{O}(a_s^2)$$

where,

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_f$$

Beta function at one-loop:

$$\beta(a_s) = -a_s^2 \beta_0 + \mathcal{O}(a_s^3)$$

RG equation:

$$\mu_R^2 \frac{da_s(\mu_R^2)}{d\mu_R^2} = -a_s^2 \beta_0 + \mathcal{O}(a_s^3)$$

Asymptotic freedom

RGE at one loop:

$$\mu_R^2 \frac{da_s}{d\mu_R^2} = -a_s^2 \beta_0$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_f$$

$$C_A = 3, \quad T_f = \frac{1}{2}$$

Integrating from initial to final scales

$$\beta_0 > 0$$

$$\frac{1}{a_s(\mu_f^2)} = \frac{1}{a_s(\mu_i^2)} + \beta_0 \log(\mu_f^2/\mu_i^2)$$

QCD Coupling constant runs!

Solution at one loop:

$$a_s(\mu_f^2) = \frac{a_s(\mu_i^2)}{1 + a_s(\mu_i^2)\beta_0 log(\mu_f^2/\mu_i^2)}$$

At large energies as $\mu_f \to \infty$

coupling vanishes $a_s(\mu_f^2) \to 0$

QCD IS ASYMPTOTICALLY FREE

At high energies QCD is weakly interacting

Beta Function of QCD

Four Loop results for beta function of QCD

$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = \beta\left(a_s(\mu_R^2)\right)$$
$$= -\beta_0\, a_s^2(\mu_R^2) - \beta_1\, a_s^3(\mu_R^2) - \beta_2\, a_s^4(\mu_R^2) - \cdots$$

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f$$

$$\beta_2 = \frac{2857}{54}C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9}C_F C_A T_F n_f - \frac{1415}{27}C_A^2 T_F n_f + \frac{44}{9}C_F T_F^2 n_f^2 + \frac{158}{27}C_A T_F^2 n_f^2$$

$$\beta_3 = C_A^4\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + C_A^3 T_F n_f\left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right)$$
$$+C_A^2 C_F T_F n_f\left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f\left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right)$$
$$+46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2\left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2\left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right)$$
$$+C_A C_F T_F^2 n_f^2\left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{424}{243}C_A T_F^3 n_f^3 + \frac{1232}{243}C_F T_F^3 n_f^3$$
$$+\frac{d_A^{abcd} d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right)$$
$$+n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)$$

Status of $\alpha_s(\mu_R)$

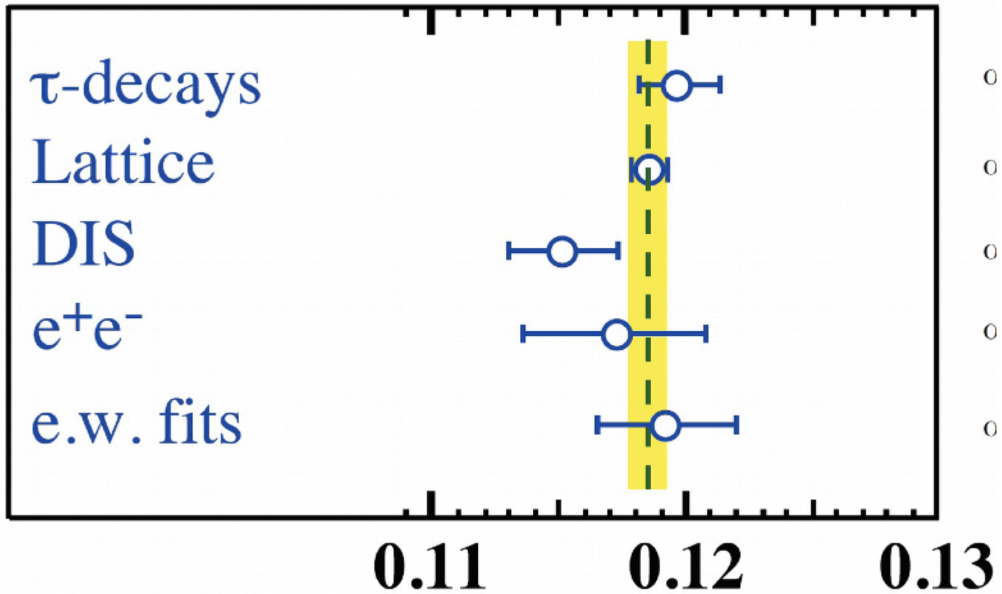
Renormalisation group equation for α_s:

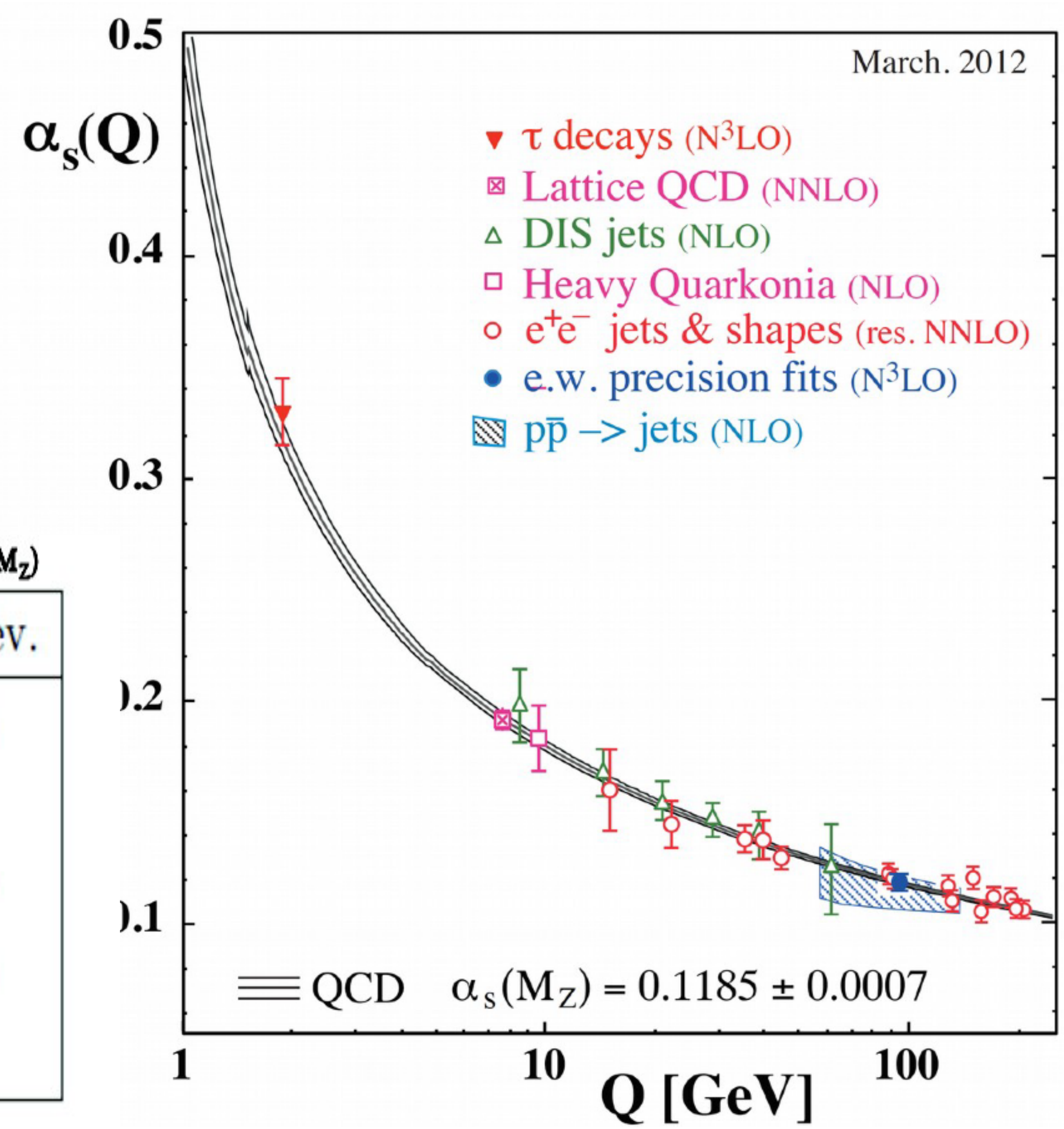
$$\mu_R^2 \frac{d}{d\mu_R^2} a_s(\mu_R^2) = \beta\left(a_s(\mu_R^2)\right)$$
$$= -\beta_0\, a_s^2(\mu_R^2) - \beta_1\, a_s^3(\mu_R^2) - \beta_2\, a_s^4(\mu_R^2) - \cdots$$

Four Loops

$\Lambda_{\overline{MS}}^{(5)} = (214 \pm 9)$ MeV
$\Lambda_{\overline{MS}}^{(4)} = (297 \pm 11)$ MeV

$\alpha_s(M_Z) = 0.1185 \pm 0.0007$





$\alpha_s(M_Z)$

Process	$\alpha_s(M_{Z^0})$	excl. mean $\alpha_s(M_{Z^0})$	std. dev.
τ-decays	0.1197 ± 0.0016	0.1183 ± 0.0007	0.8
Lattice QCD	0.1186 ± 0.0007	0.1182 ± 0.0011	0.3
DIS $[F_2]$	0.1151 ± 0.0022	0.1188 ± 0.0010	1.5
e^+e^- [jets & shps]	0.1172 ± 0.0037	0.1185 ± 0.0006	0.3
ew. prec. data]	0.1192 ± 0.0028	0.1185 ± 0.0006	0.2

Quantum Chromodynamics

The Nobel Prize in Physics 2004

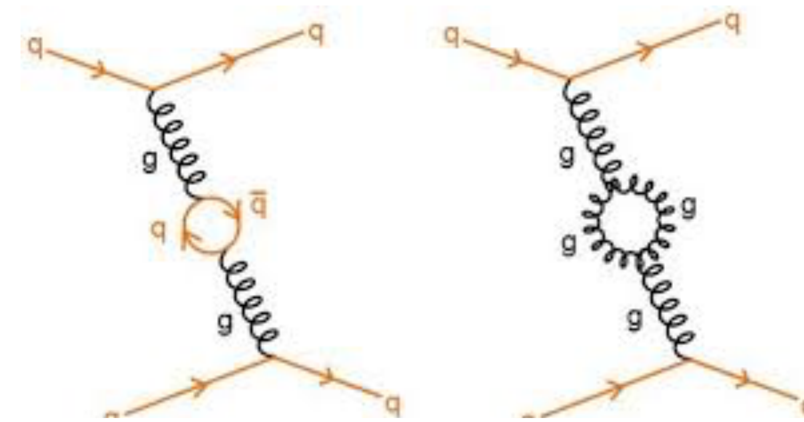
"for the discovery of asymptotic freedom in the theory of the strong interaction"

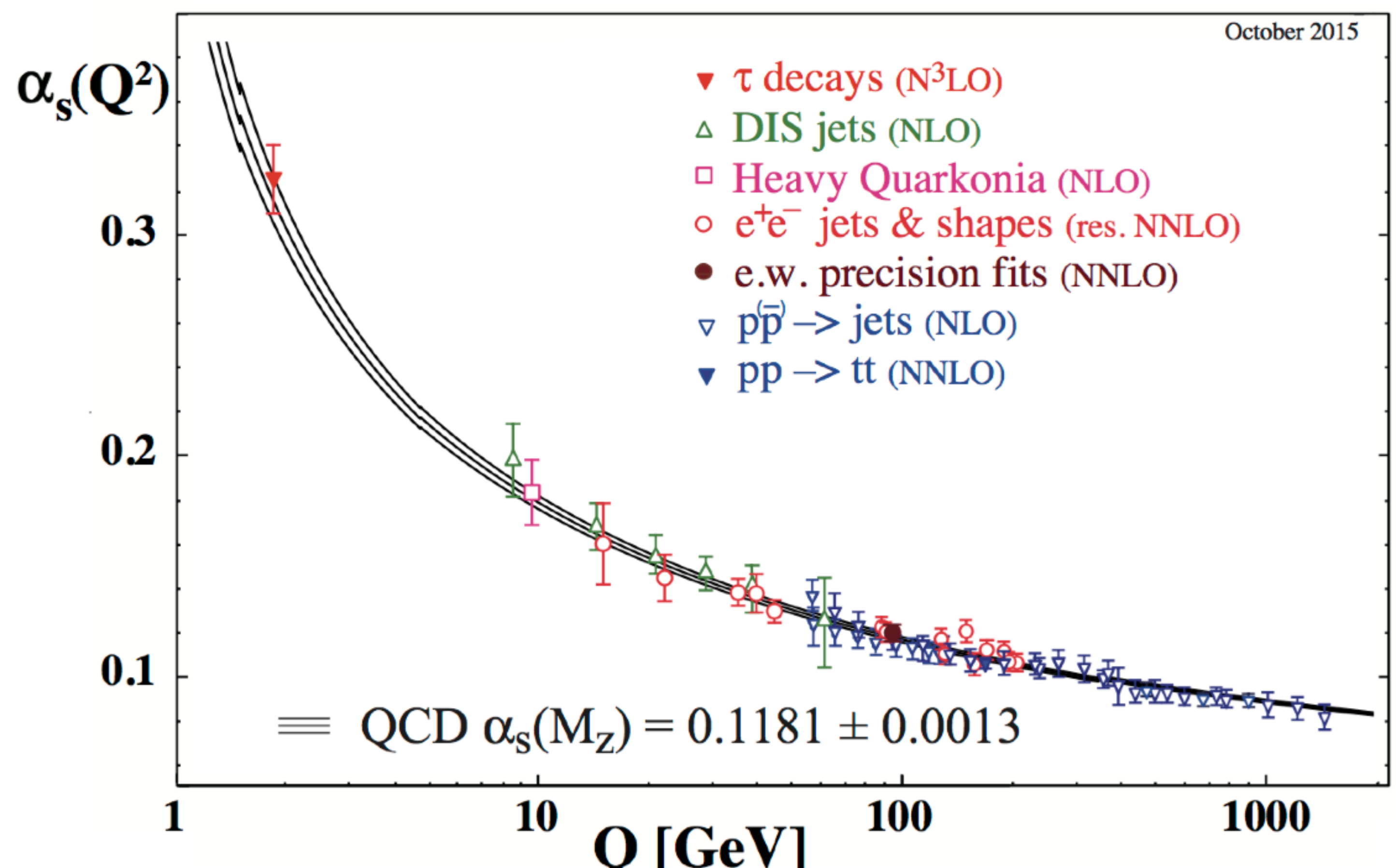
David J. Gross | **H. David Politzer** | **Frank Wilczek**

Asymptotic Freedom

$$Q^2 \to \infty \qquad \alpha_s(Q^2) \to 0$$

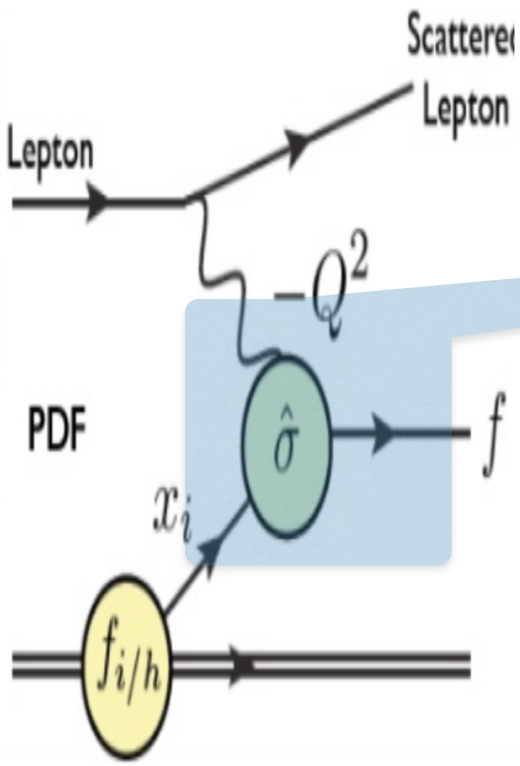
Non-Abelian Gauge theory - SU(3)

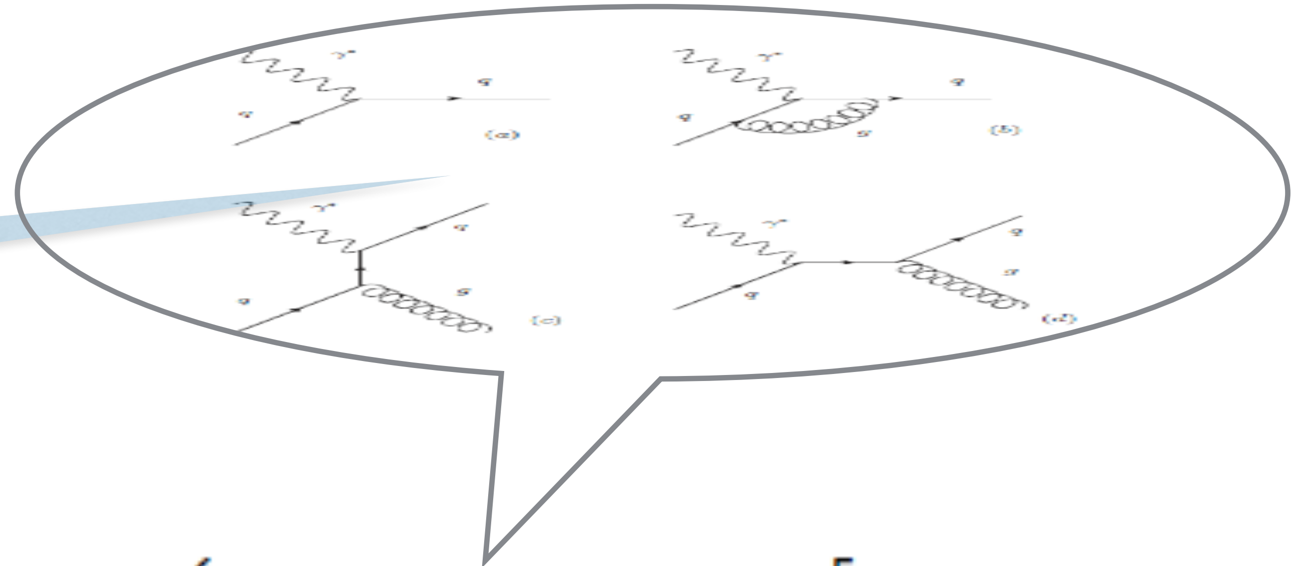


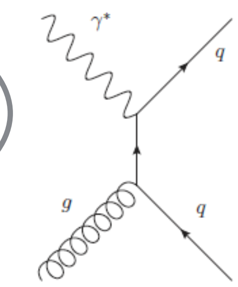


Accommodates Bjorken Scaling

QCD improved Parton Model







$$F_2(x,Q^2) = x\sum_{q,\bar{q}} e_q^2 \left\{ q_0(x) + \frac{\alpha_s}{2\pi}\int_0^1 \frac{dz}{z} q_0\left(\frac{x}{z}\right)\left[P(z)\log\left(\frac{Q^2}{m^2}\right) + C(z)\right] + \ldots \right\}$$

Collinear Renormalisation

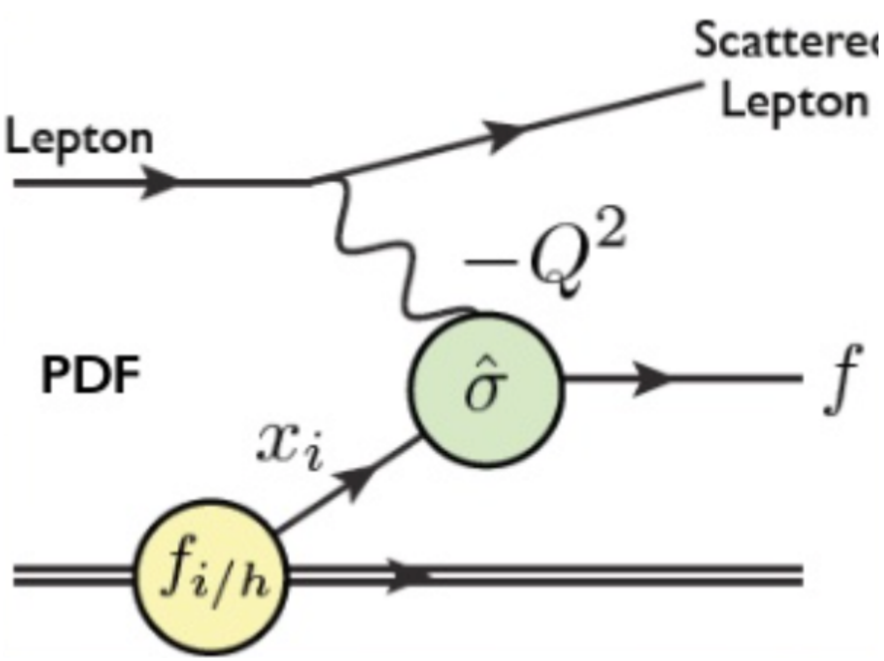
Factorisation Scale

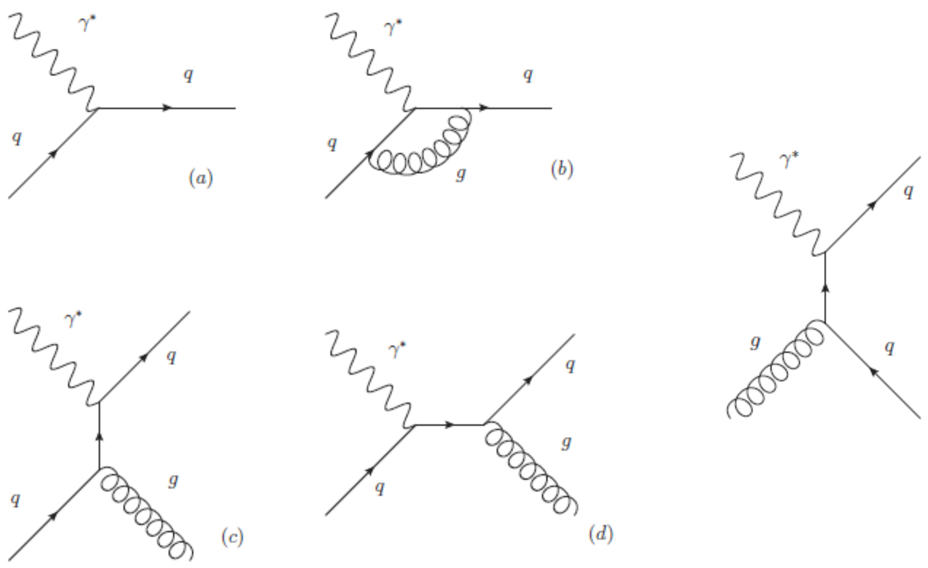
$$q(x,\mu) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} q_0\left(\frac{x}{z}\right)\left[P(z)\log\left(\frac{\mu^2}{m^2}\right) + C(z)\right] + \ldots$$

$$\log\frac{Q^2}{m^2} = \log\frac{Q^2}{\mu^2} + \log\frac{\mu^2}{m^2}.$$

$$\begin{aligned} F_2(x,Q^2) &= x\sum_{q\bar{q}} e_q^2 \int_x^1 \frac{dz}{z} q\left(\frac{x}{z},Q^2\right)\left[\delta(1-z) + \frac{\alpha_s}{2\pi} C_q^{\overline{MS}}(z) + \ldots\right] \\ &\quad + x\sum_{q\bar{q}} e_q^2 \int_x^1 \frac{dz}{z} g\left(\frac{x}{z},Q^2\right)\left[\frac{\alpha_s}{2\pi} C_g^{\overline{MS}}(z) + \ldots\right] \end{aligned}$$

Scaling Violation





μ_F - Factorisation Scale

μ_R - Renormalisation Scale

$$\sigma^P(x,Q^2) = \sum_{i=q,\bar{q},g} \int_x^1 \frac{dz}{z} C_i\left(z,Q^2,\mu_R^2,\mu_F^2\right) f_{i/P}\left(\frac{x}{z},\mu_F^2\right)$$

Process Dependent Coefficient function
Perturbqtively Calculable to all orders

Only Parton and Target dependent
Non-Perturbative

Scaling Violation

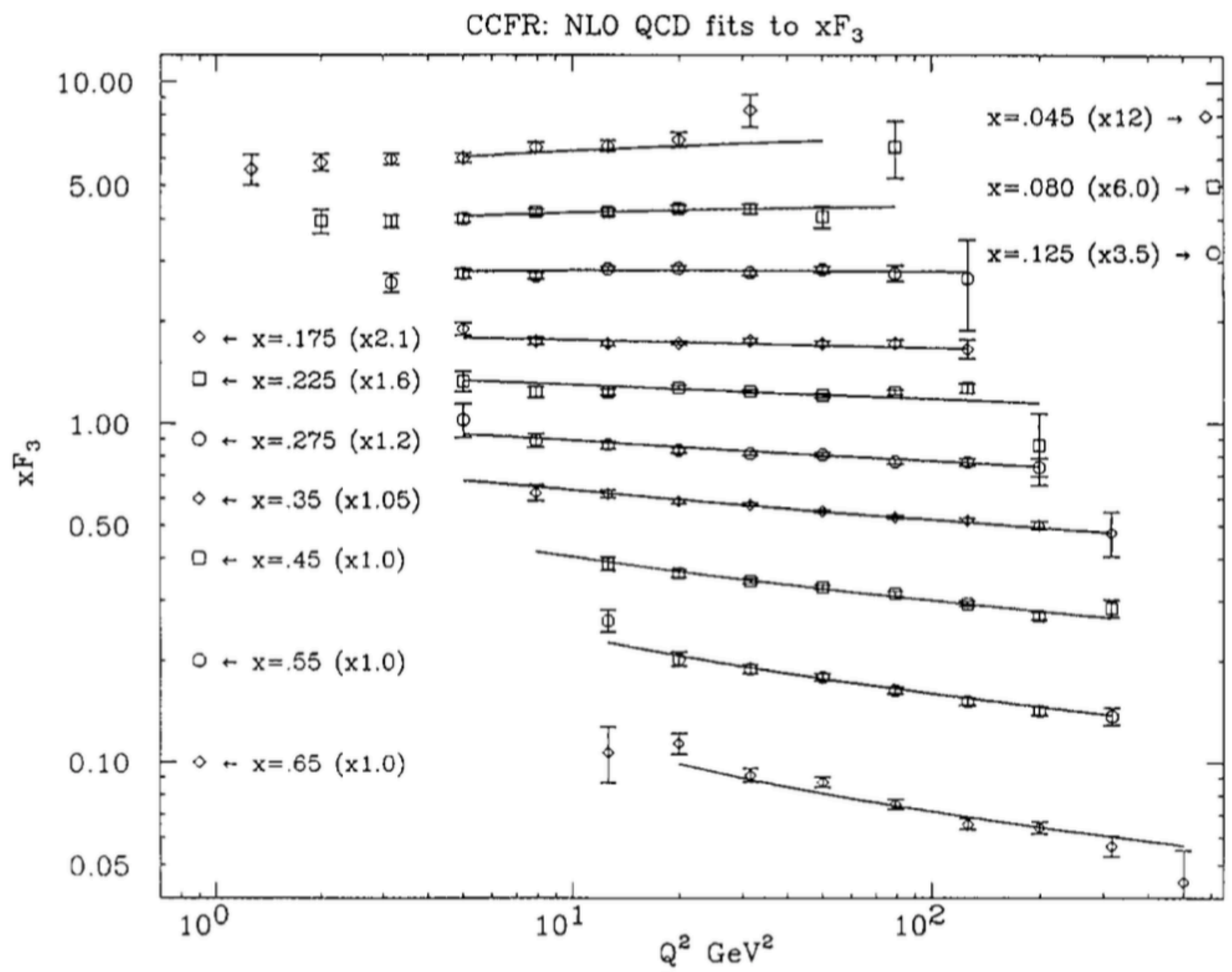


Figure 5.7.b. The xF_3 data and the best NLO QCD fit. Cuts of $Q^2 > 5$ GeV^2 and $x < 0.7$ were applied for a next-to-leading order fit including target mass corrections.

DGLAP Evolution

Collinear Renormalisation

$$q(x,\mu) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} q_0\left(\frac{x}{z}\right)\left[P(z)\log\left(\frac{\mu^2}{m^2}\right) + C(z)\right] + \ldots$$

Arbitrariness in the choice of $\mu = \mu_F$

$$\mu^2 \frac{d}{d\mu^2} q_0(z) = 0$$

Collinear Renormalisation Group Equation

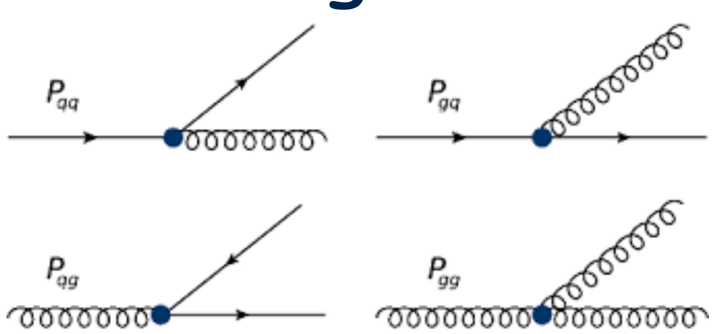
DGLAP Evolution Equation

$$\frac{\partial}{\partial \log \mu^2}\begin{pmatrix} q_i \\ g \end{pmatrix}(x,\mu^2) = \frac{\alpha_s}{2\pi}\int_x^1 \sum_{j=q,\bar{q}} \frac{d\xi}{\xi}\begin{pmatrix} P_{ij}\left(\frac{x}{\xi},\alpha_s\right) & P_{ig}\left(\frac{x}{\xi},\alpha_s\right) \\ P_{gj}\left(\frac{x}{\xi},\alpha_s\right) & P_{gg}\left(\frac{x}{\xi},\alpha_s\right) \end{pmatrix}\begin{pmatrix} q_j \\ g \end{pmatrix}(\xi,\mu^2).$$

In QCD perturbation

$$P_{ij}^{N^mLO}(x,\mu^2) = \sum_{k=0}^{m} a_s^{k+1}(\mu^2) P_{ij}^{(k)}(x).$$

Leading Order

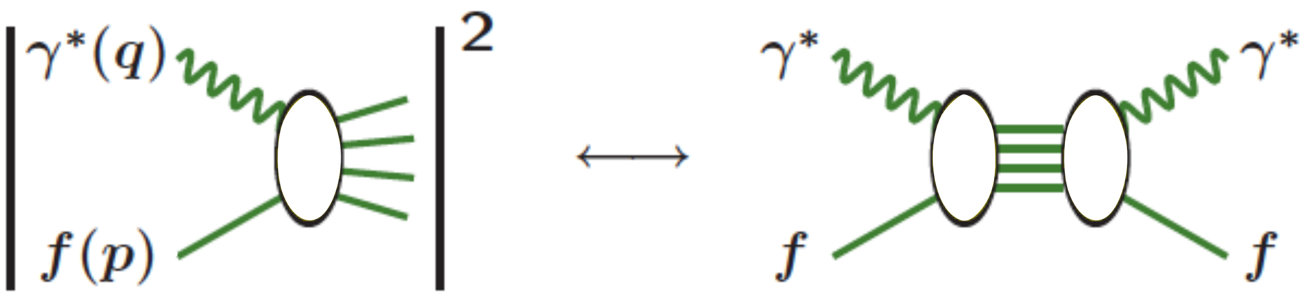


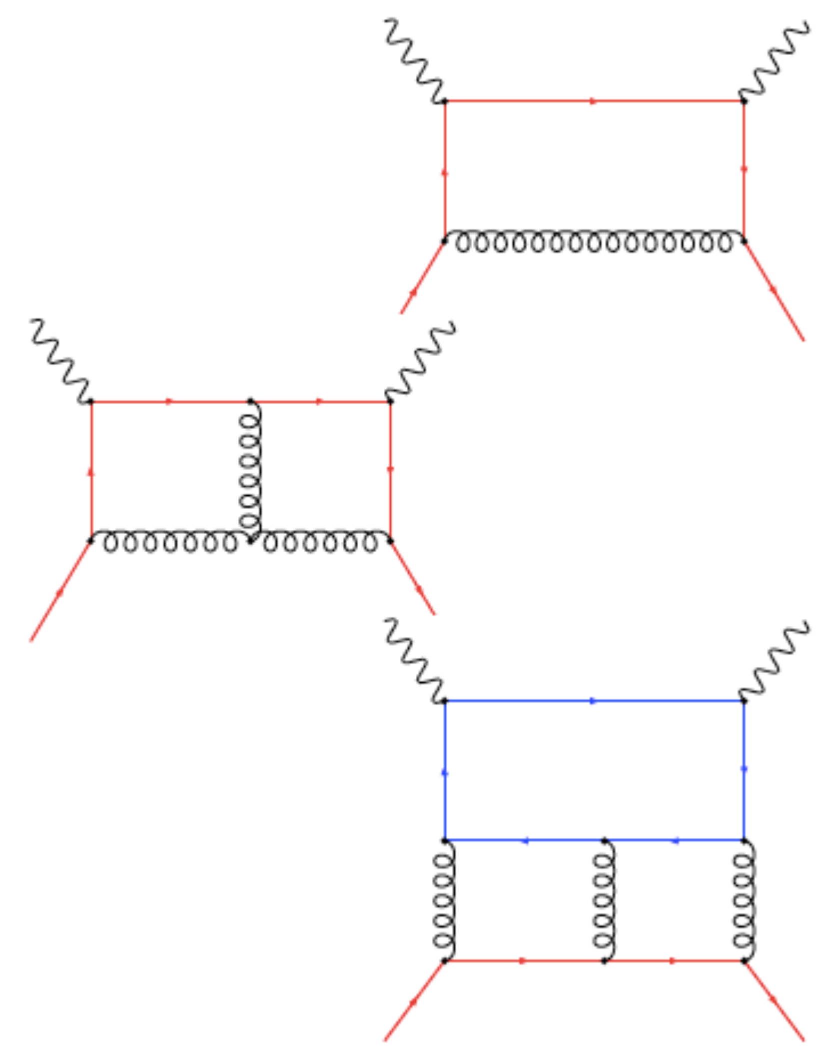
NNLO Results

[Moch,Vogt, Vermaseren]

Optical Theorem

UV + IR Poles in Dim. Regularisation





Poles in Dim. Regularisation

$$
\begin{aligned}
P_{ij}(x,\mu^2) &= a_s(\mu^2)P_{ij}^{(0)}(x) \\
&+ a_s^2(\mu^2)P_{ij}^{(1)}(x) \\
&+ a_s^3(\mu^2)P_{ij}^{(2)}(x)
\end{aligned}
$$

Finite part

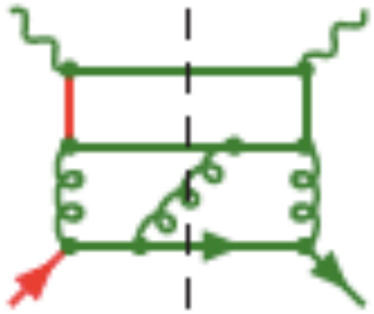
$$C_j(x,\mu^2) = C_j^{(0)}(x,\mu^2) + a_s(\mu^2)C_j^{(1)}(x,\mu^2) + a_s^2(\mu^2)C_j^{(2)}(x,\mu^2)$$

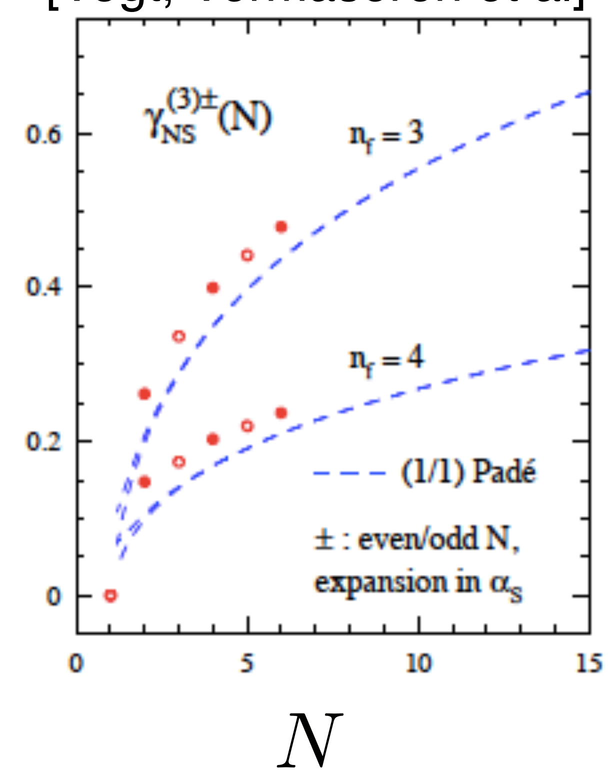
Going beyond NNLO

[Vogt, Vermaseren et al]

MINCER to **FORCER**

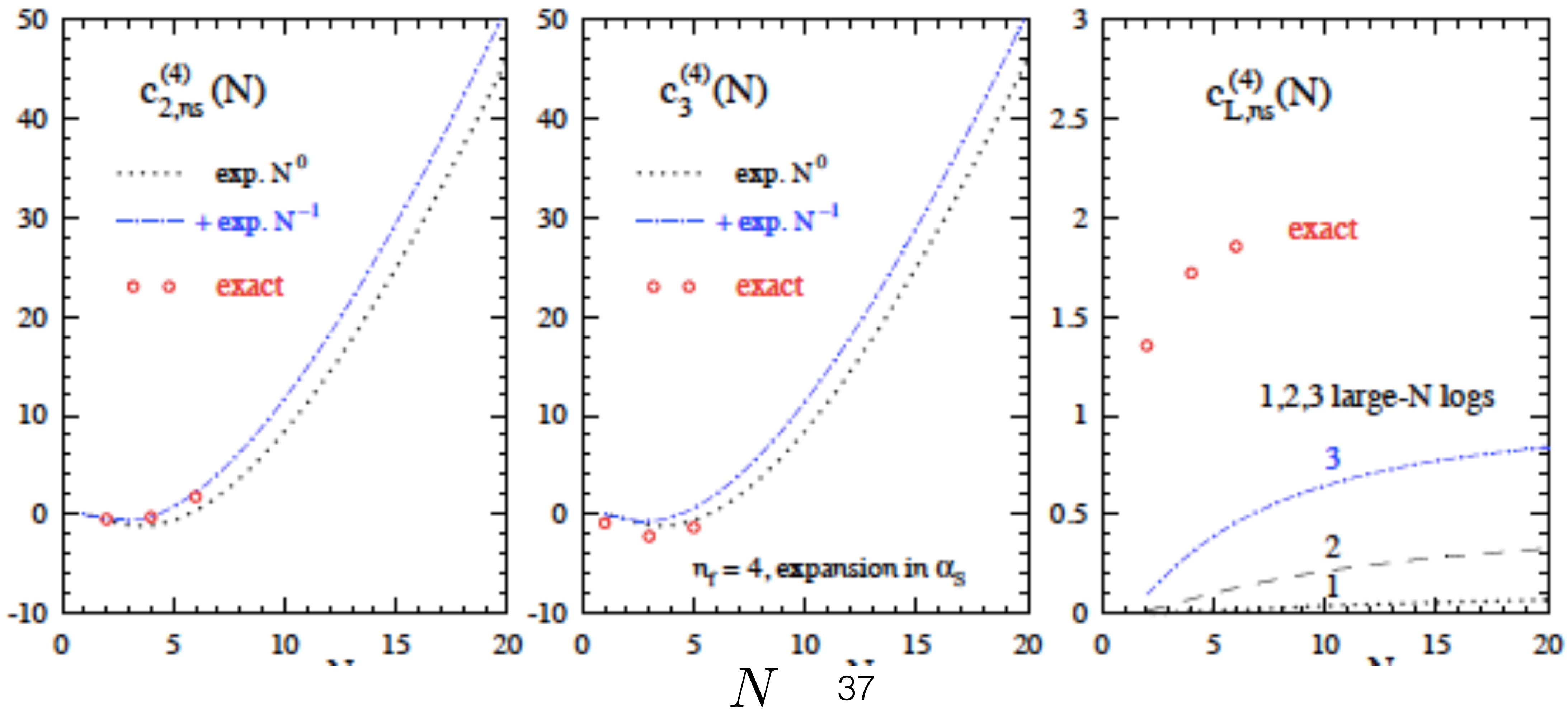
for 4 loop results





Third order contributions to Coefficient and splitting functions

$C_{2,3,L}^{(3)}$ $\quad P_{ij}^{(3)}(x), \gamma_{ij}^{(3)}(N)$



Parametrisation of PDFs

Standard form

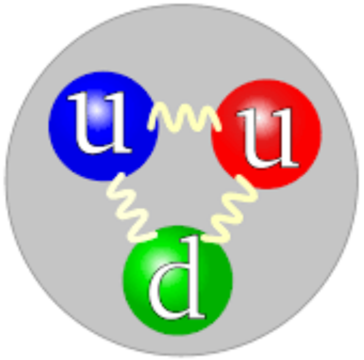
at initial scale μ_0

$$f(x, \mu_o^2) = \alpha_o x^{\alpha_1} (1-x)^{\alpha_2} P(x)$$

where $P(x) = (1 + \alpha_3 x + \alpha_4 x^2 + \cdots) e^{\beta_1 x} (1 + e^{\beta_4} x)^{\beta_5}$

Simple Constraints

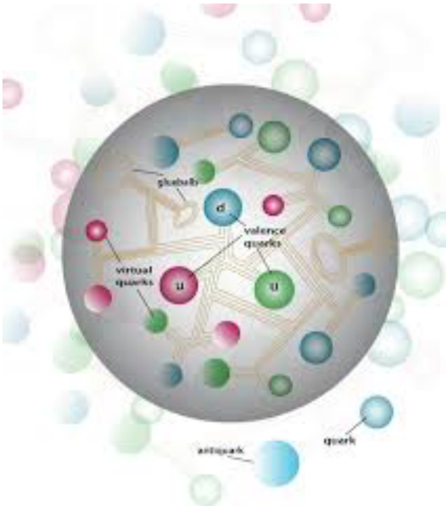
$$\int_0^1 dx \, (f_{u/P}(x, \mu^2) - f_{\bar{u}/P}(x, \mu^2)) = 2 \qquad \int_0^1 dx \left(f_{d/P}(x, \mu^2) - f_{\bar{d}/P}(x, \mu^2) \right) = 1$$



$$\int_0^1 dx \, (f_{s/P}(x, \mu^2) - f_{\bar{s}/P}(x, \mu^2)) = 0$$

Momentum sum rule

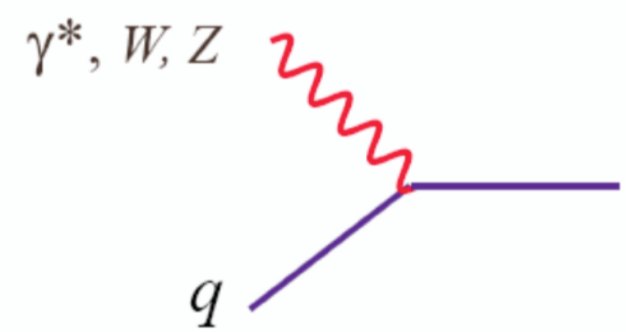
$$\int_0^1 dx \; x \left(\sum_i (f_{q_i/P}(x, \mu^2) - f_{\bar{q}_i/P}(x, \mu^2)) + f_{g/P}(x, \mu^2) \right) = 1$$



Observables for PDF exactraction

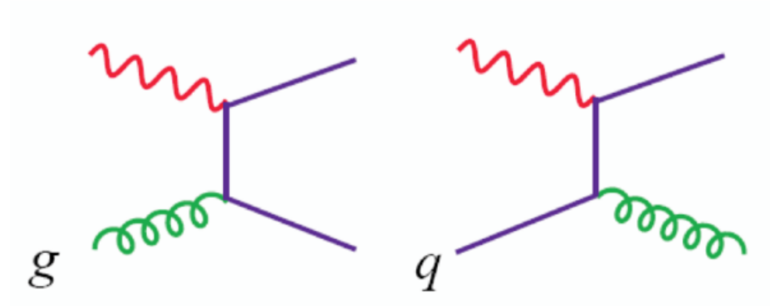
$DIS - eN, \mu N$

(CDHS,CHARM,CCFR,CHORUS,NuTeV)



$DIS - \nu N, \overline{\nu} N$

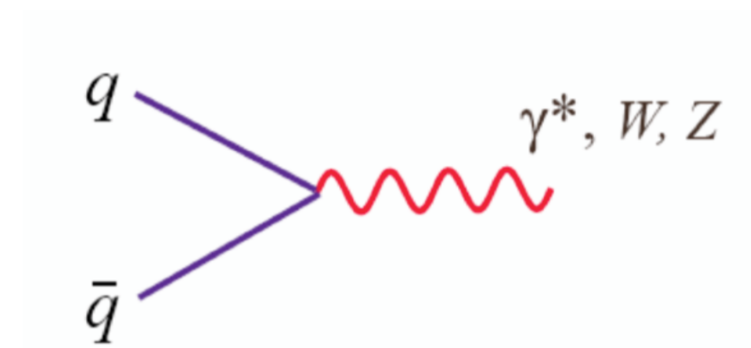
(SLAC,BCDMS,NMC,E665,H1,ZEUS)



$p\overline{p} \to jets$

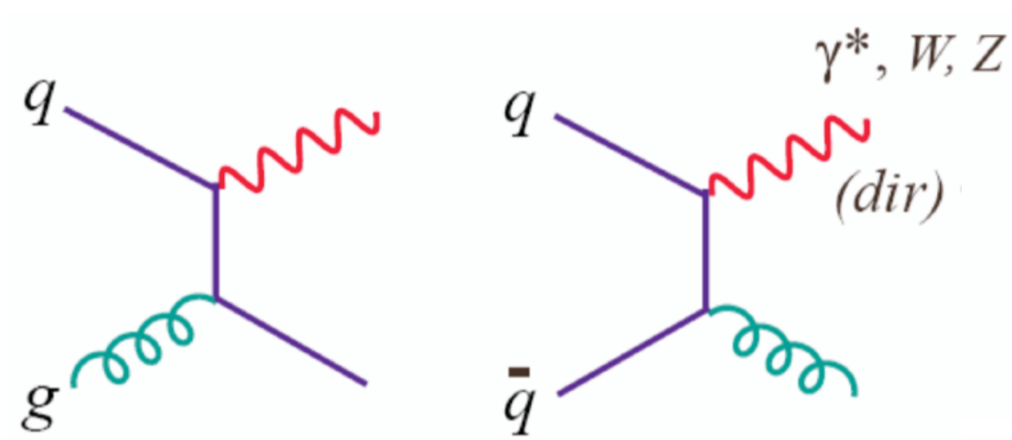
(CDF,D0)

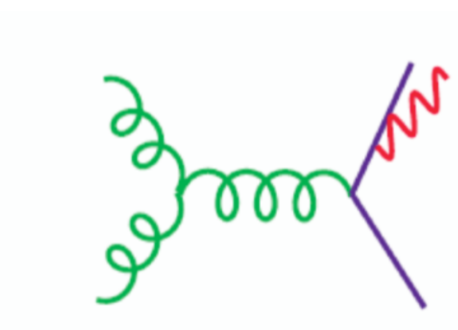
Drell-Yan



Prompt photon

(WA70,UA6,E706)





PDF extraction

GRV, GJR ...

MRST, MSTW ...

CTEQ, CT# ...

NNPDF

ABM, ABKM

Index of /archive/lhapdf/pdfsets/6.1

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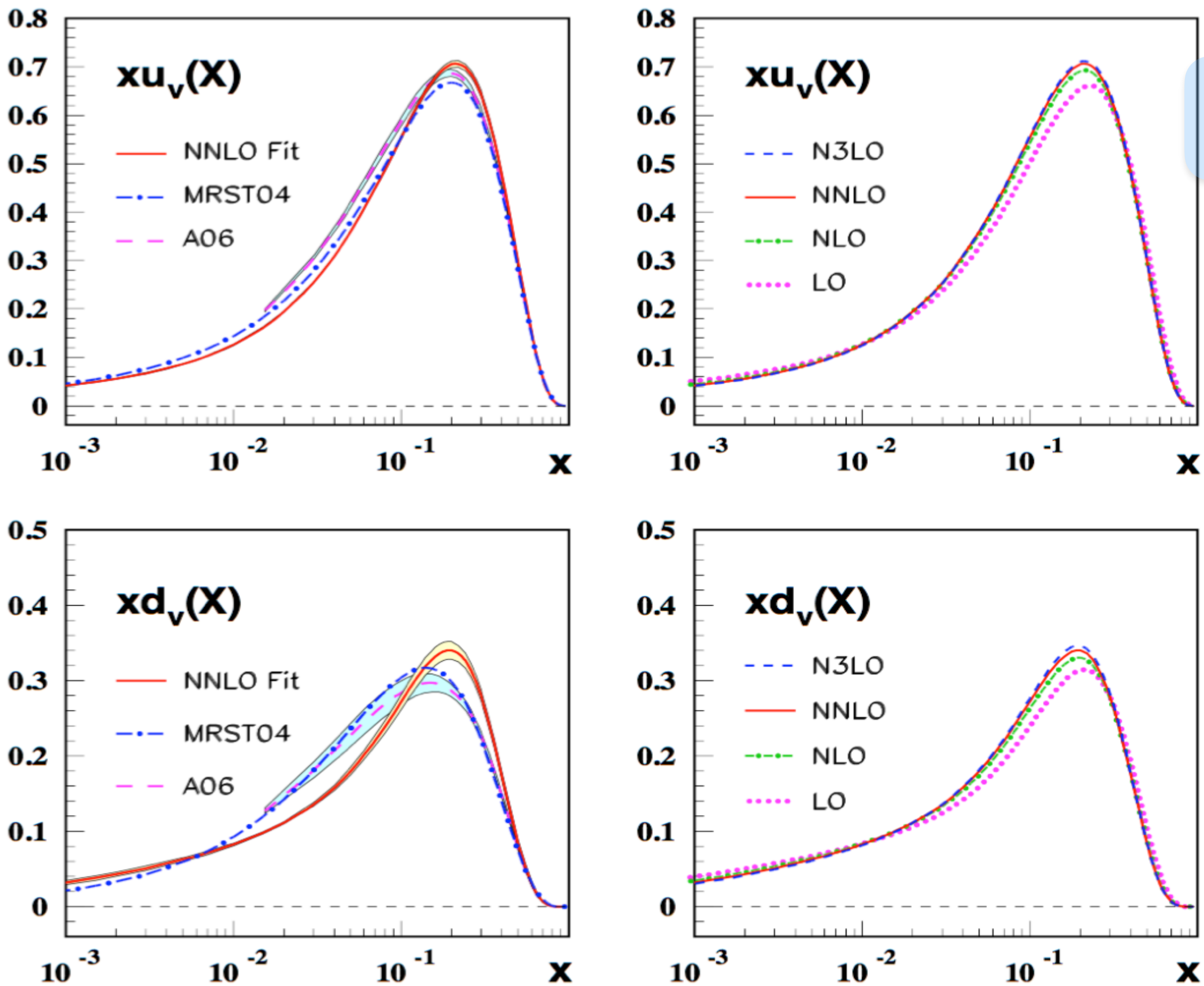
Long List of 19 pages

PDFs at approx. N3LO

[Bluemlein et al]

World Data: NS-analysis

W^2> 12.5 GeV^2 , Q^2 > 4 GeV^2

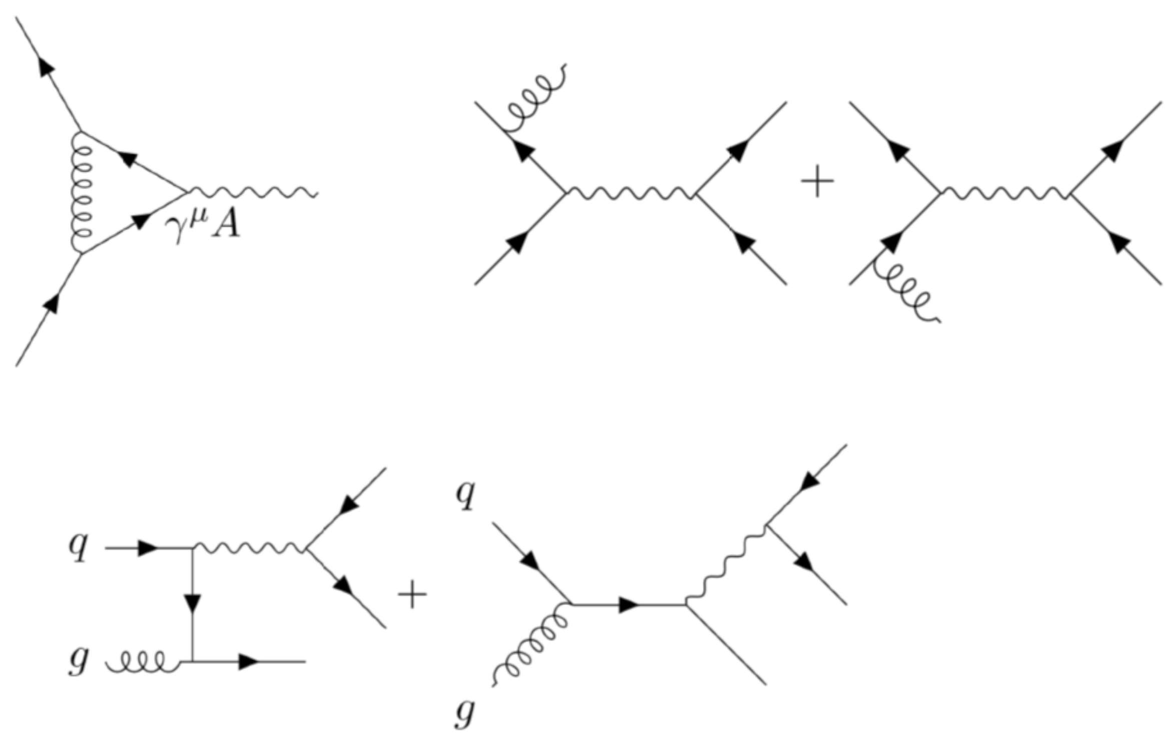


$$\alpha_s(M_Z) = 0.1145 \pm 0.0009 \text{ (exp.)}$$

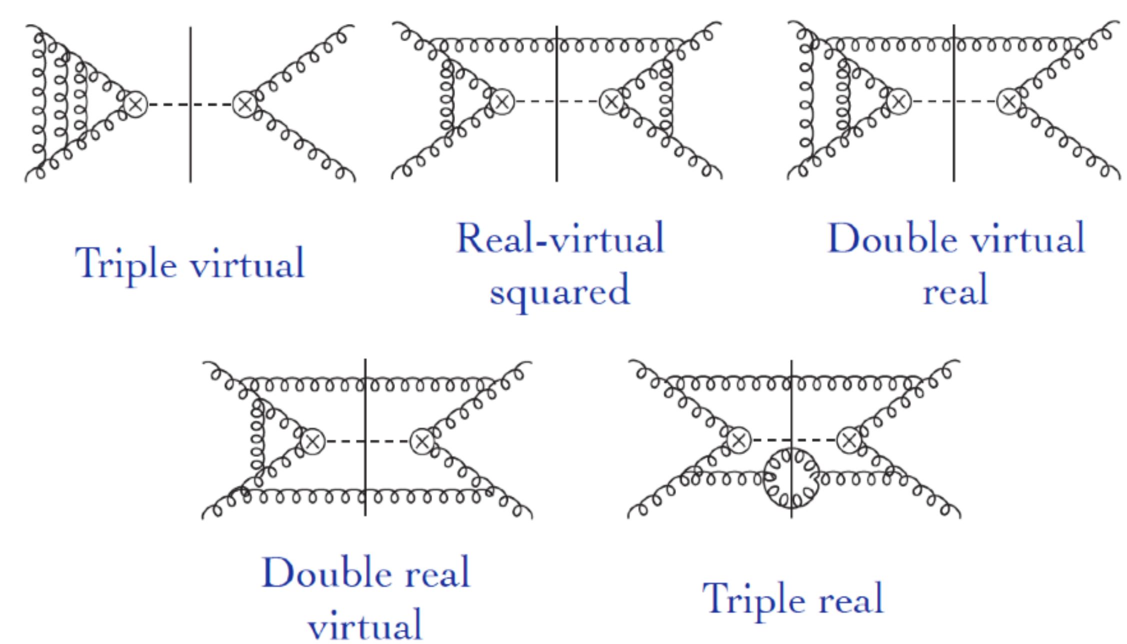
DY and Higgs production:

Coefficient function $\Delta(z,q^2) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\Delta^{(i)}(z,q^2,\mu_R^2)$

Drell-Yan Production



Higgs Production

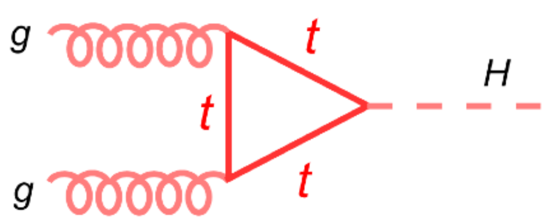


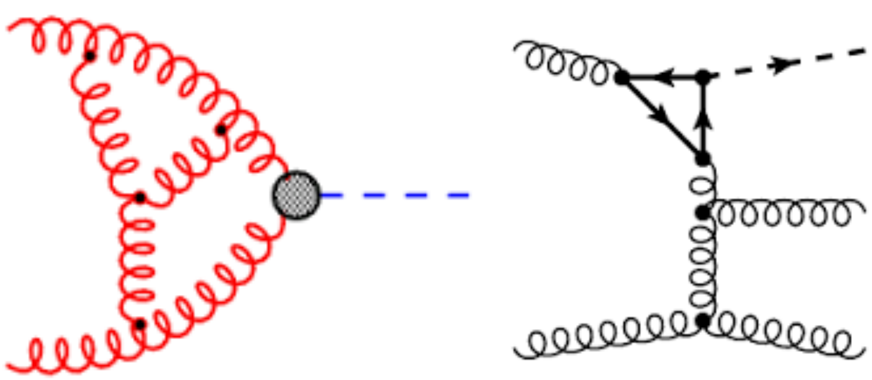
Inclusive Higgs production

Anastasiou, Duhr, Dulat and Mistleberger (`19)

$$\hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_S \hat{\sigma}^{NLO}(z) + \alpha_S^2 \hat{\sigma}^{NNLO}(z) + \alpha_S^3 \hat{\sigma}^{N3LO}(z) + \mathcal{O}(\alpha_S^4)$$

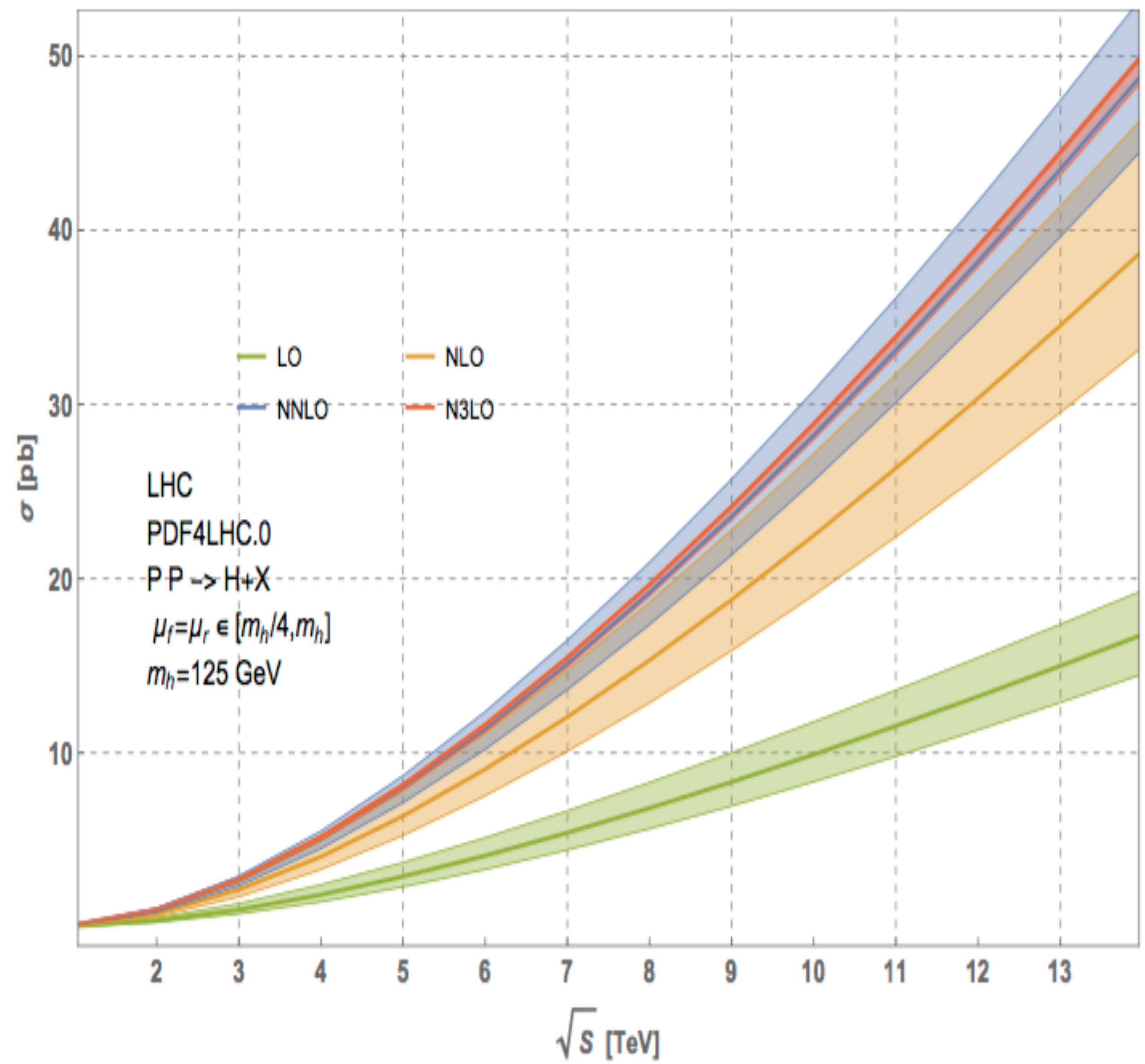
✓ ✓ ✓ ✓





LO	$15.05 \pm 14.8\%$
NLO	$38.2 \pm 16.6\%$
NNLO	$45.1 \pm 8.8\%$
N3LO	$45.2 \pm 1.9\%$

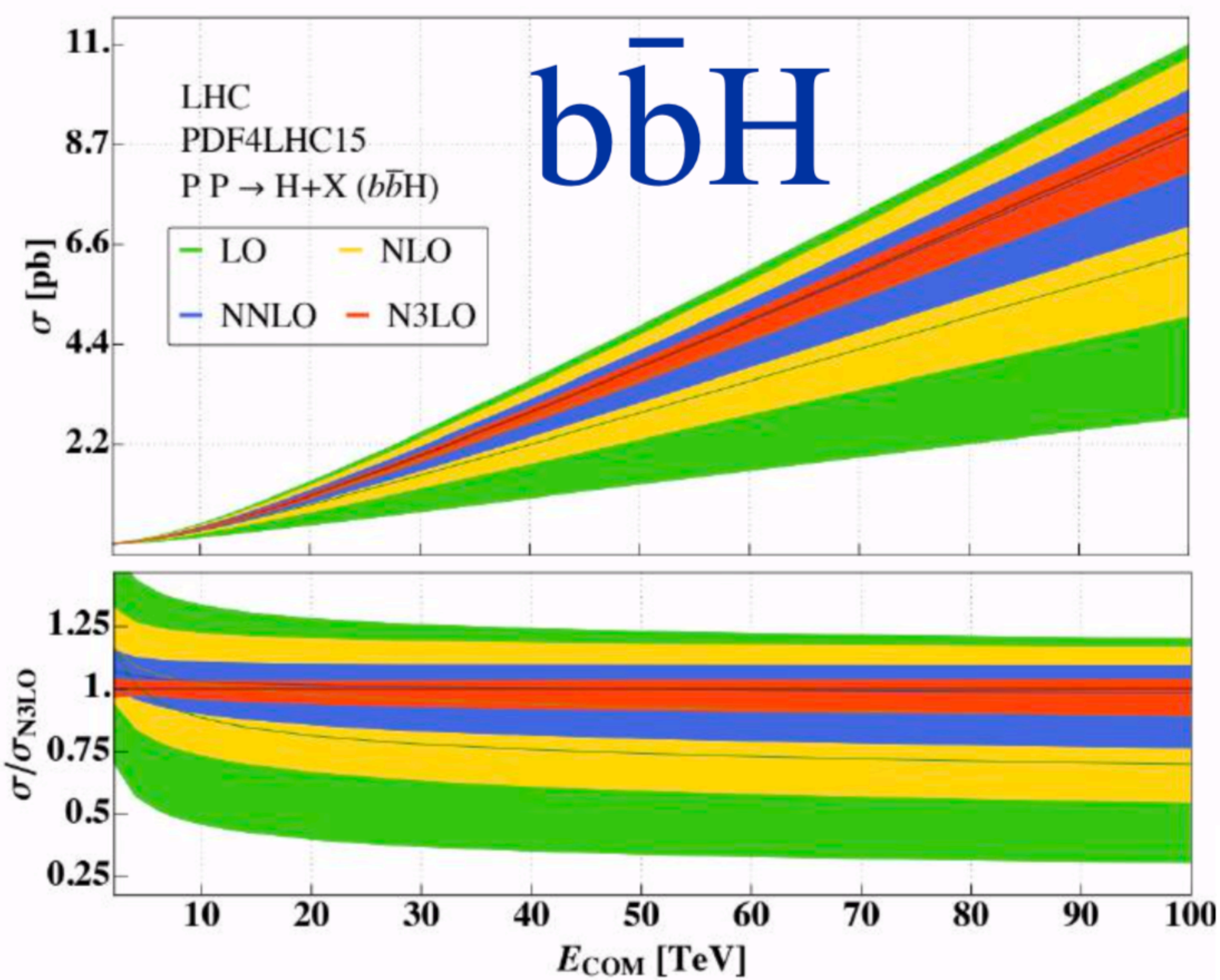
pb



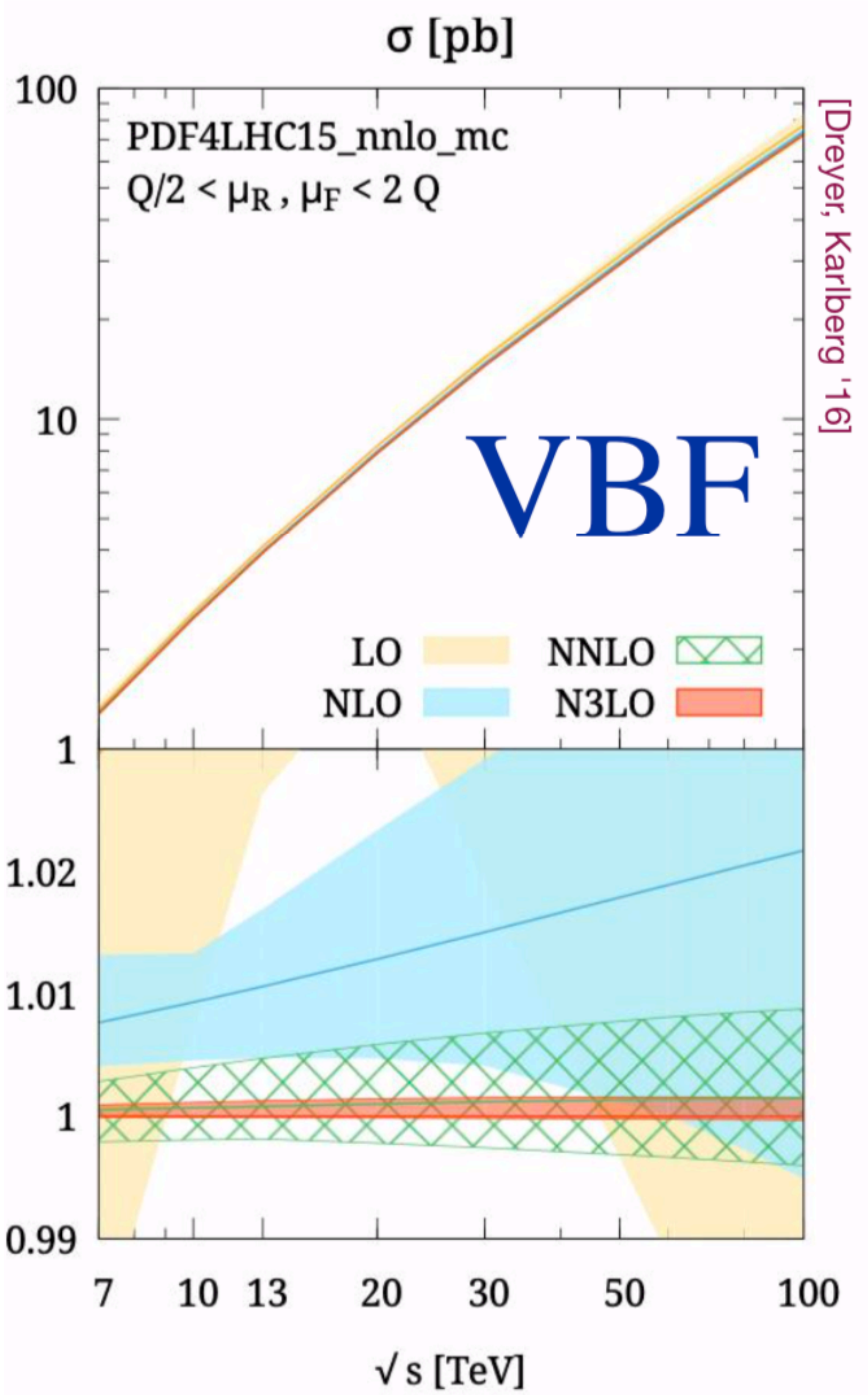
Inclusive Higgs production

$$\hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_S \hat{\sigma}^{NLO}(z) + \alpha_S^2 \hat{\sigma}^{NNLO}(z) + \alpha_S^3 \hat{\sigma}^{N3LO}(z) + \mathcal{O}(\alpha_S^4)$$

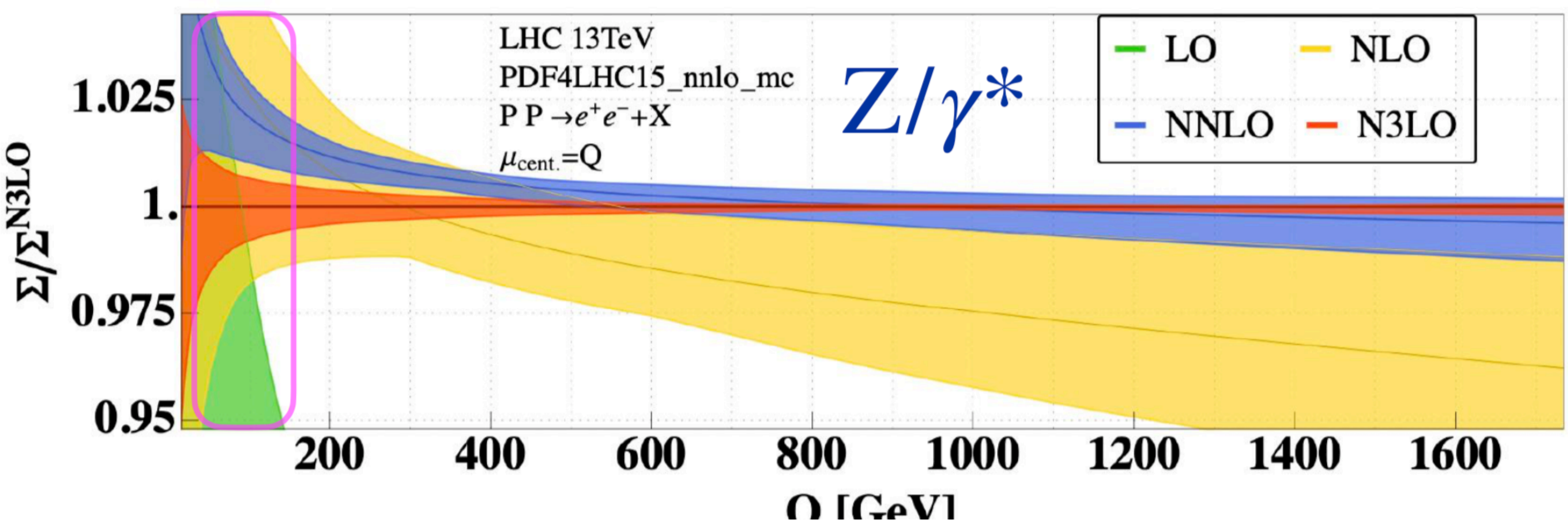
[Dulat, Duhr, Mistlberger '19]

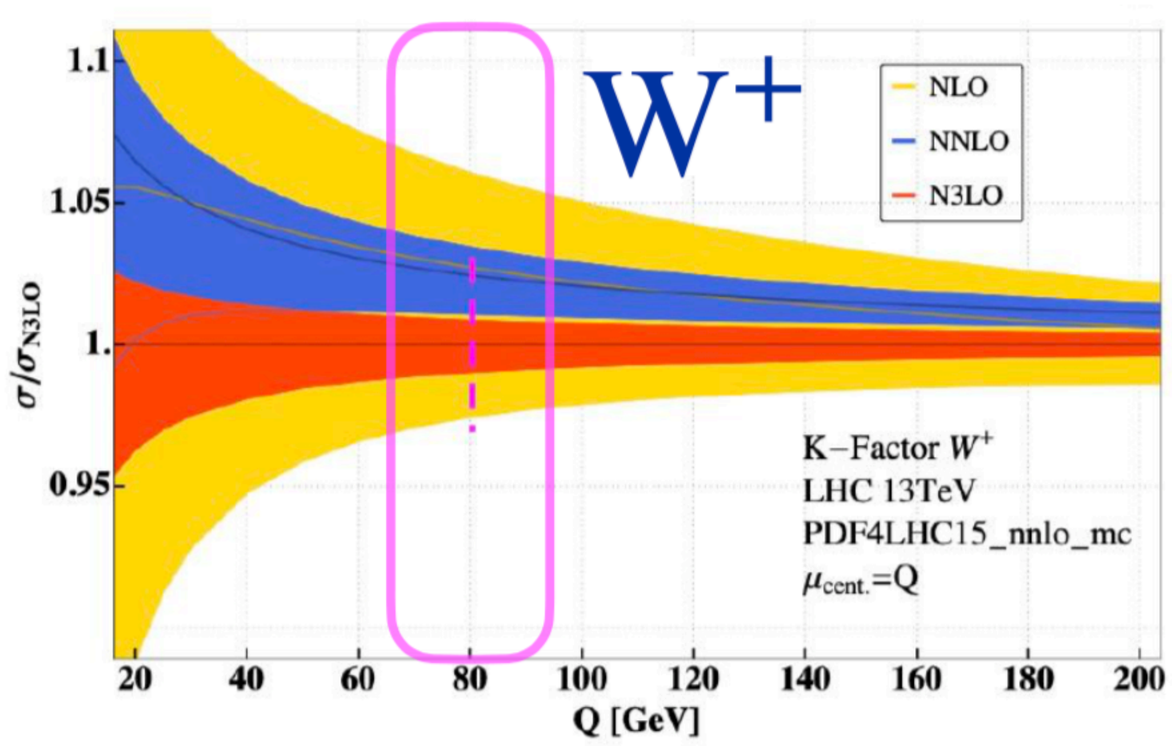


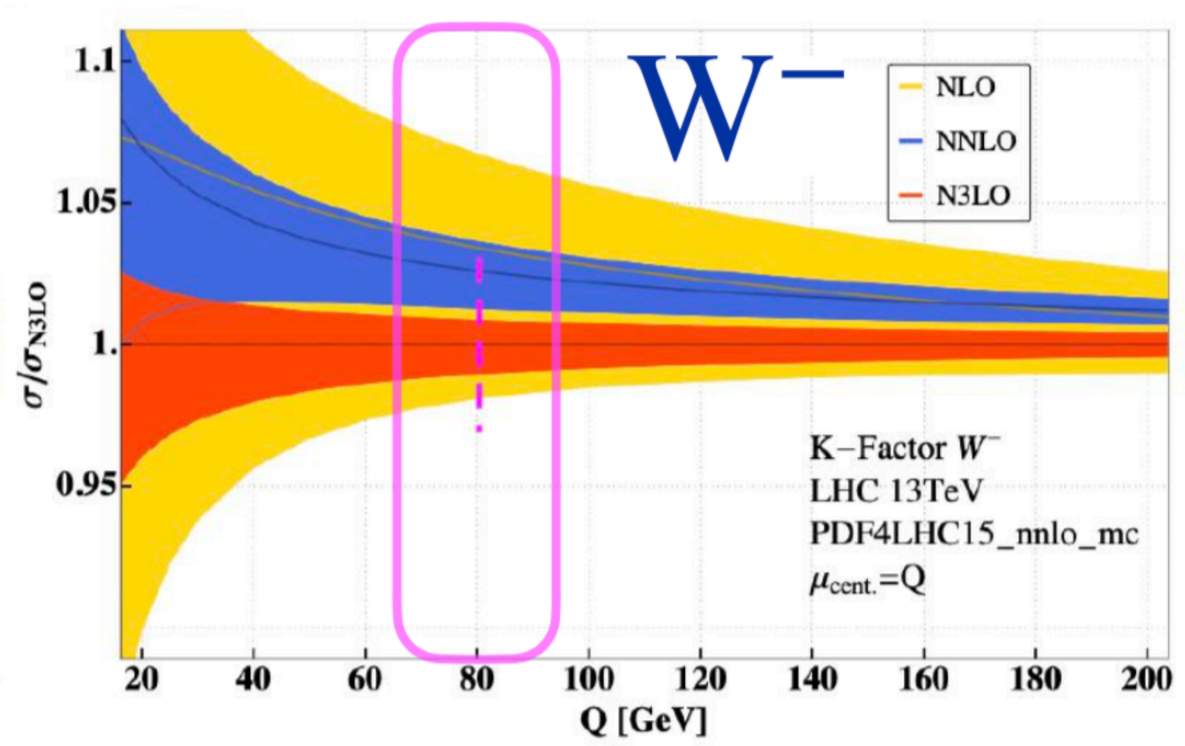
Inclusive Higgs production



Drell-Yan production





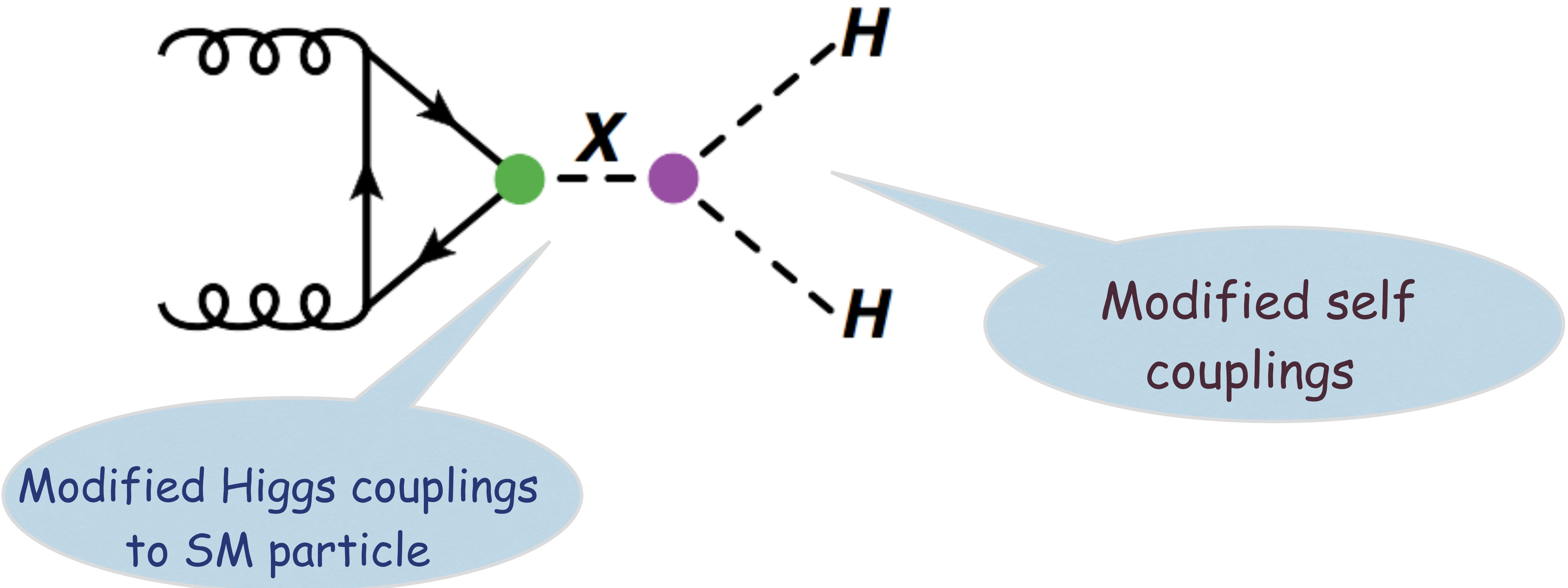


Inclusive results

Snowmass: “The Path forward to N[3]LO” [Caola, Chen, Duhr, Liu, Mistlberger, Petriello, Vita, Weinzierl '22]

	Q [GeV]	K-factor	δ(scale) [%]	$\delta(\text{PDF} + \alpha_S)$	δ(PDF-TH)
$gg \to$ Higgs	m_H	1.04	$^{+0.21\%}_{-2.37\%}$	$\pm 3.2\%$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	0.978	$^{+3.0\%}_{-4.8\%}$	$\pm 8.4\%$	$\pm 2.5\%$
NCDY	30	0.952	$^{+1.53\%}_{-2.54\%}$	$^{+3.7\%}_{-3.8\%}$	$\pm 2.8\%$
	100	0.979	$^{+0.66\%}_{-0.79\%}$	$^{+1.8\%}_{-1.9\%}$	$\pm 2.5\%$
CCDY(W^+)	30	0.953	$^{+2.5\%}_{-1.7\%}$	$\pm 3.95\%$	$\pm 3.2\%$
	150	0.985	$^{+0.5\%}_{-0.5\%}$	$\pm 1.9\%$	$\pm 2.1\%$
CCDY(W^-)	30	0.950	$^{+2.6\%}_{-1.6\%}$	$\pm 3.7\%$	$\pm 3.2\%$
	150	0.984	$^{+0.6\%}_{-0.5\%}$	$\pm 2\%$	$\pm 2.13\%$

In BSM scenarios



- *Non-Resonant production*

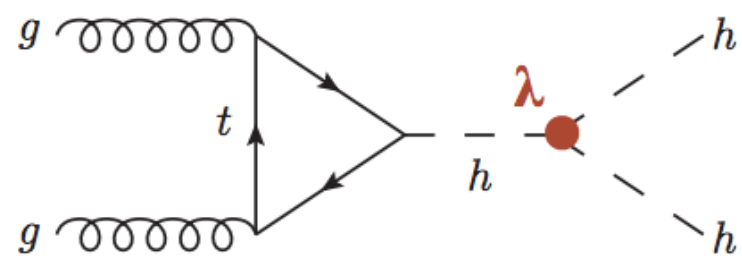
- *Resonant production:*

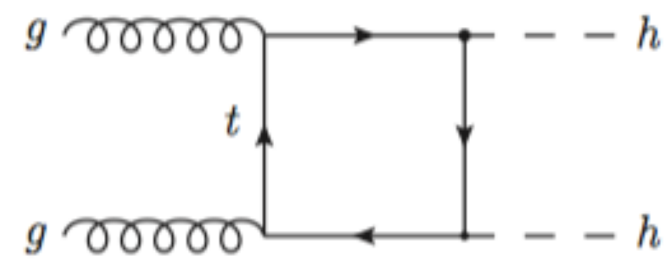
Heavy scalars in Two Higgs doublet models,
Spin-2 resonances from Randal Sundrum Model

Production Cross section

- Dominant ones:

Glover,van der Bij





- Relative Contributions

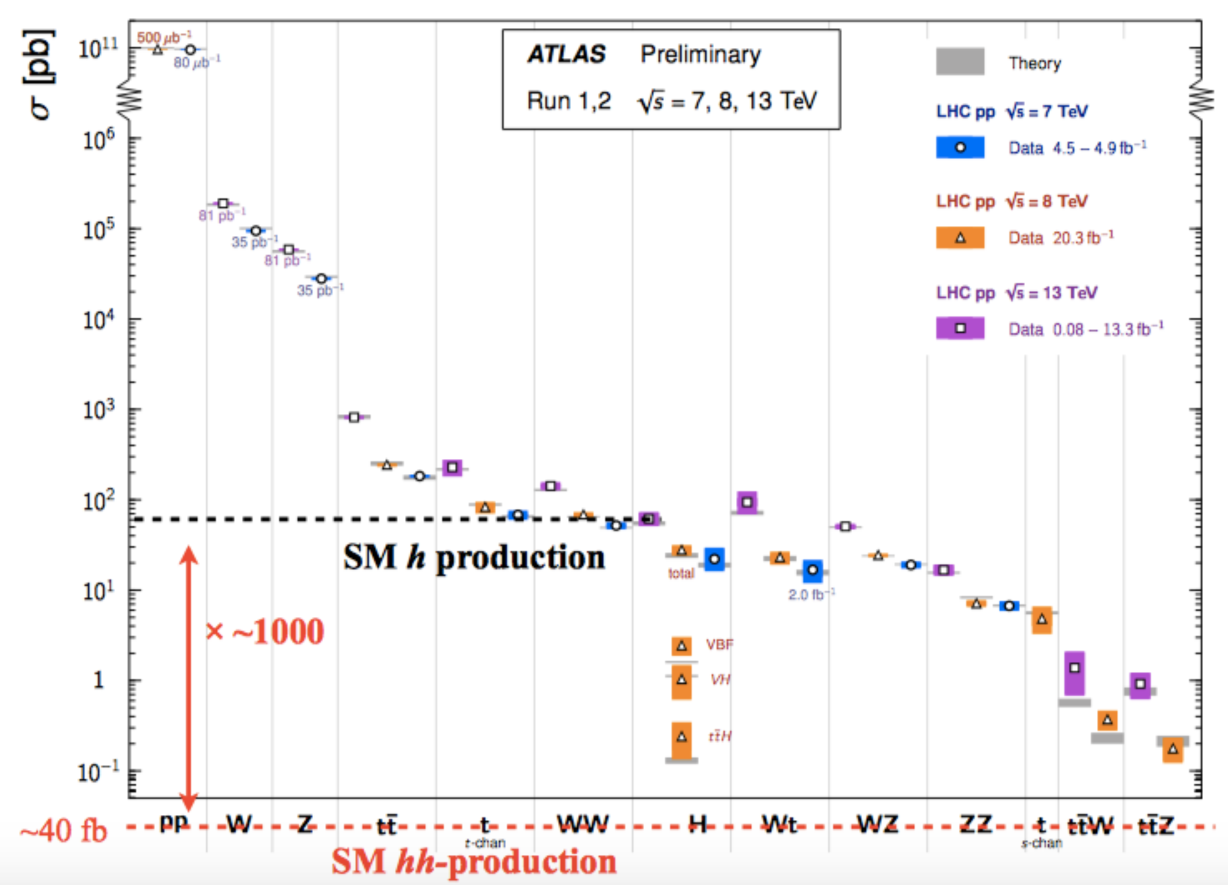
destructively interfere!

$\lambda_3^{SM} = 0.3$

ggF-hh	$\sim$ 40 fb
VBF-hh	~2 fb
V-hh	$\sim$ 1 fb
tt-hh	$\sim$ 1 fb

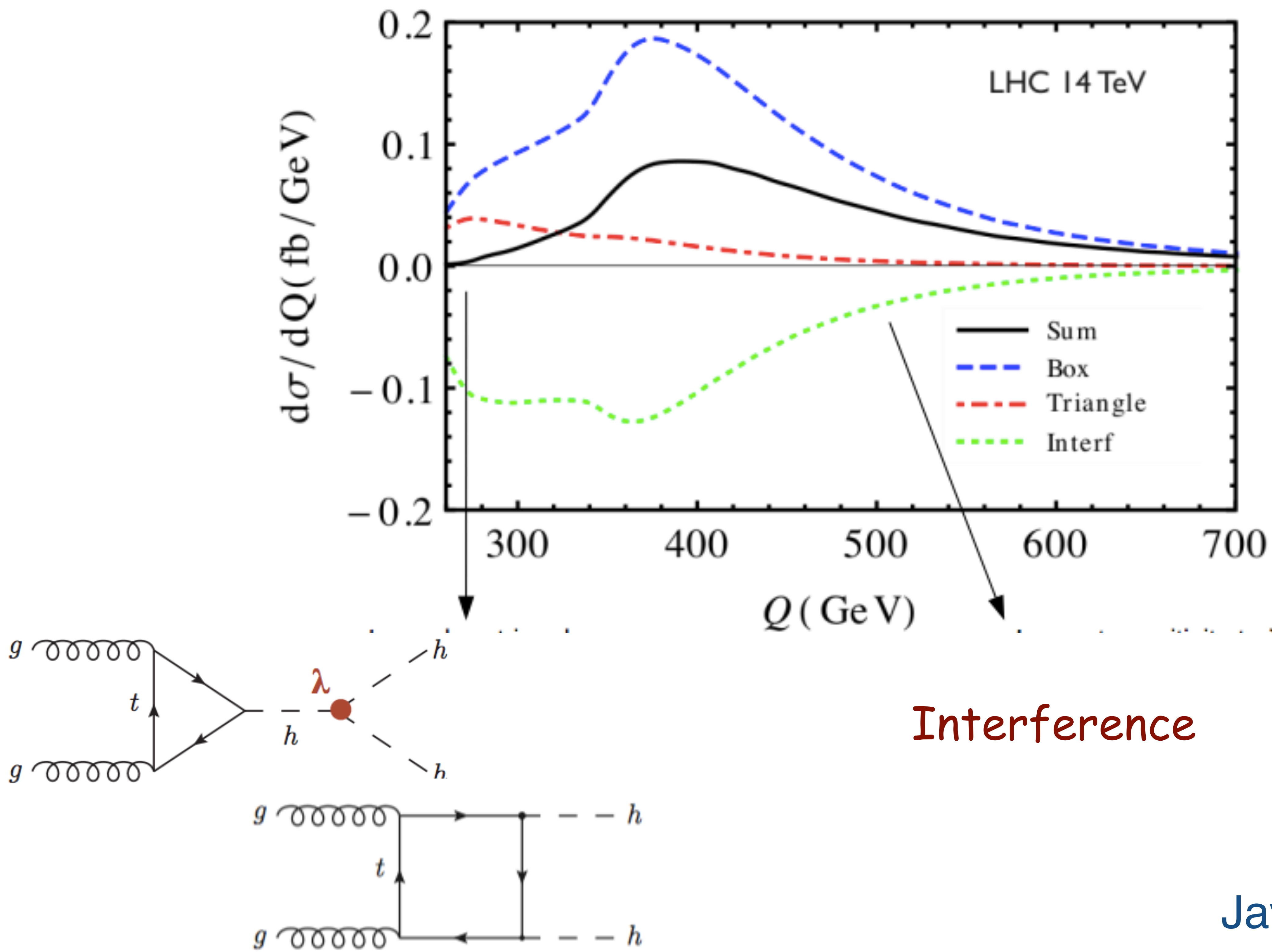
$$b + \overline{b} \rightarrow hh \quad \approx 0.1 fb$$

Tough Task



Dominant Production

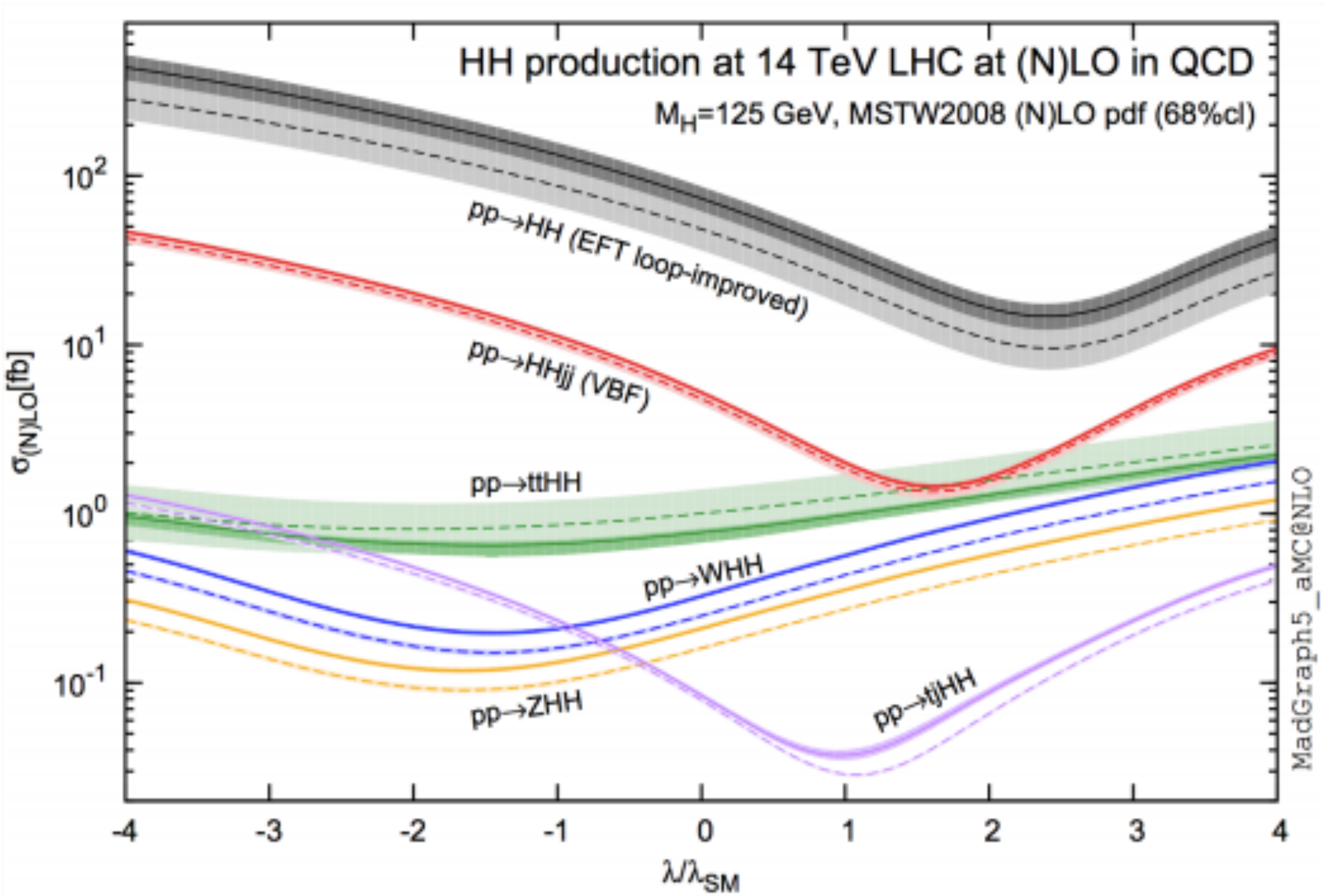
- Dominant channel at LO: $g + g \to hh$



Javier 2018

Relative contributions at NLO

- Di-Higgs boson at NLO in QCD

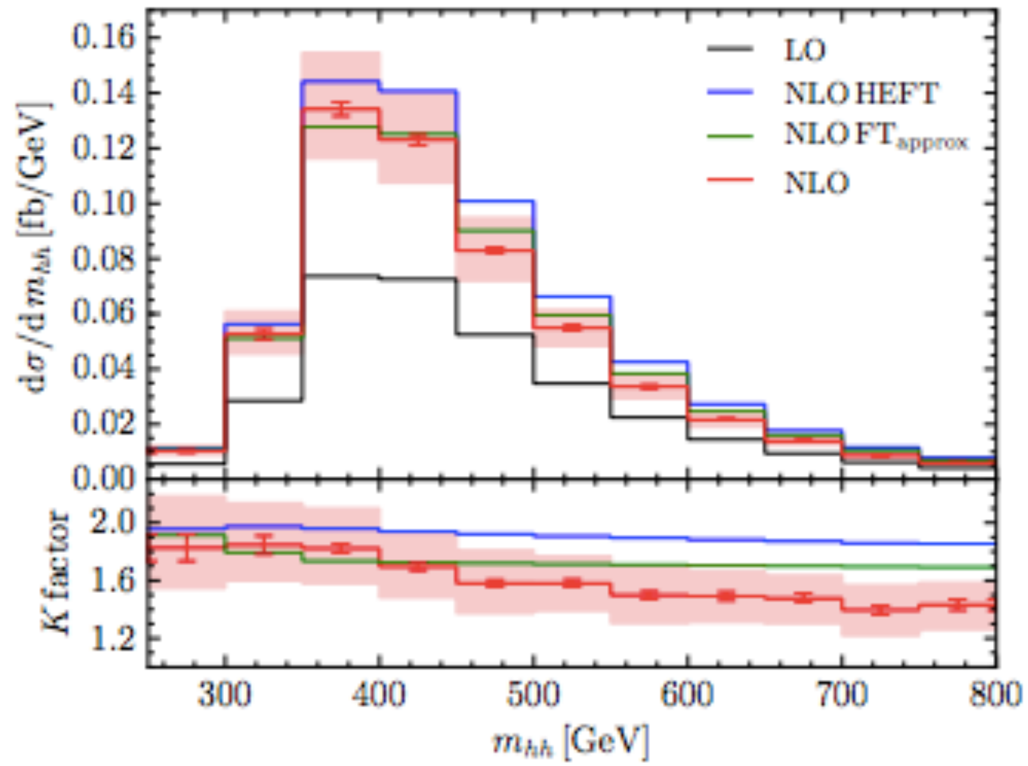


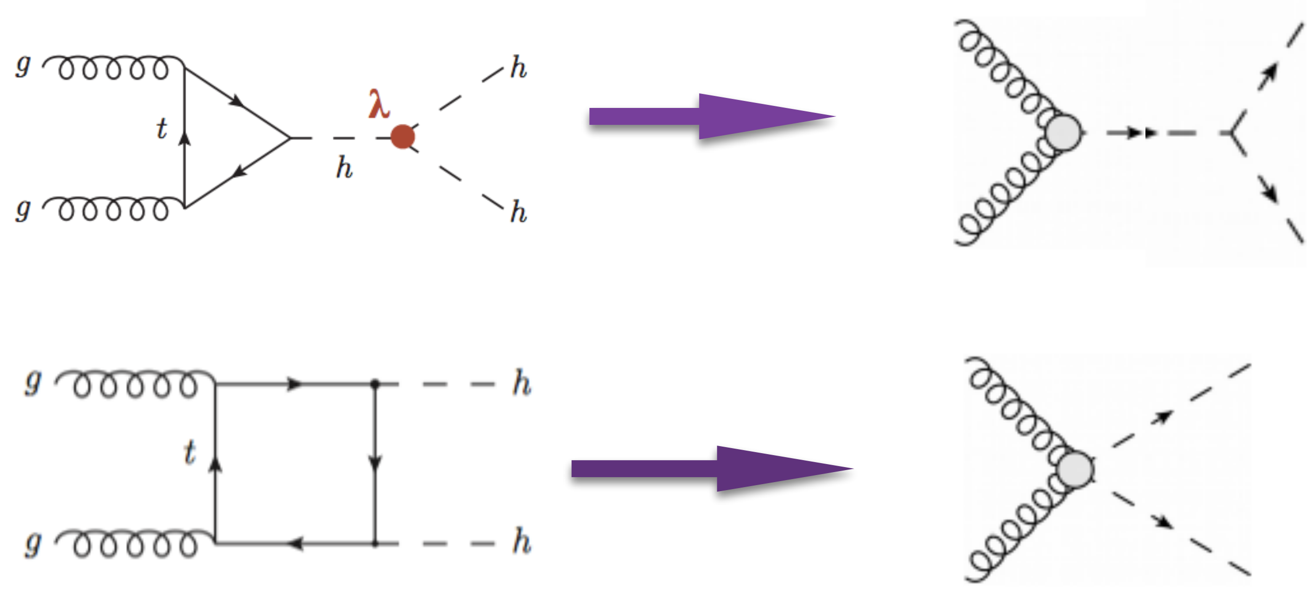
Beyond LO from gluon fusion

Exact top mass

Effective Field theory
$m_t \to \infty$

$$\sigma^{\rm NLO}(pp \to hh) = \sigma^{\rm LO} + \sigma^{\rm virt} + \sum_{i,j \in \{g,q,\bar{q}\}} \sigma^{\rm real}_{ij}$$



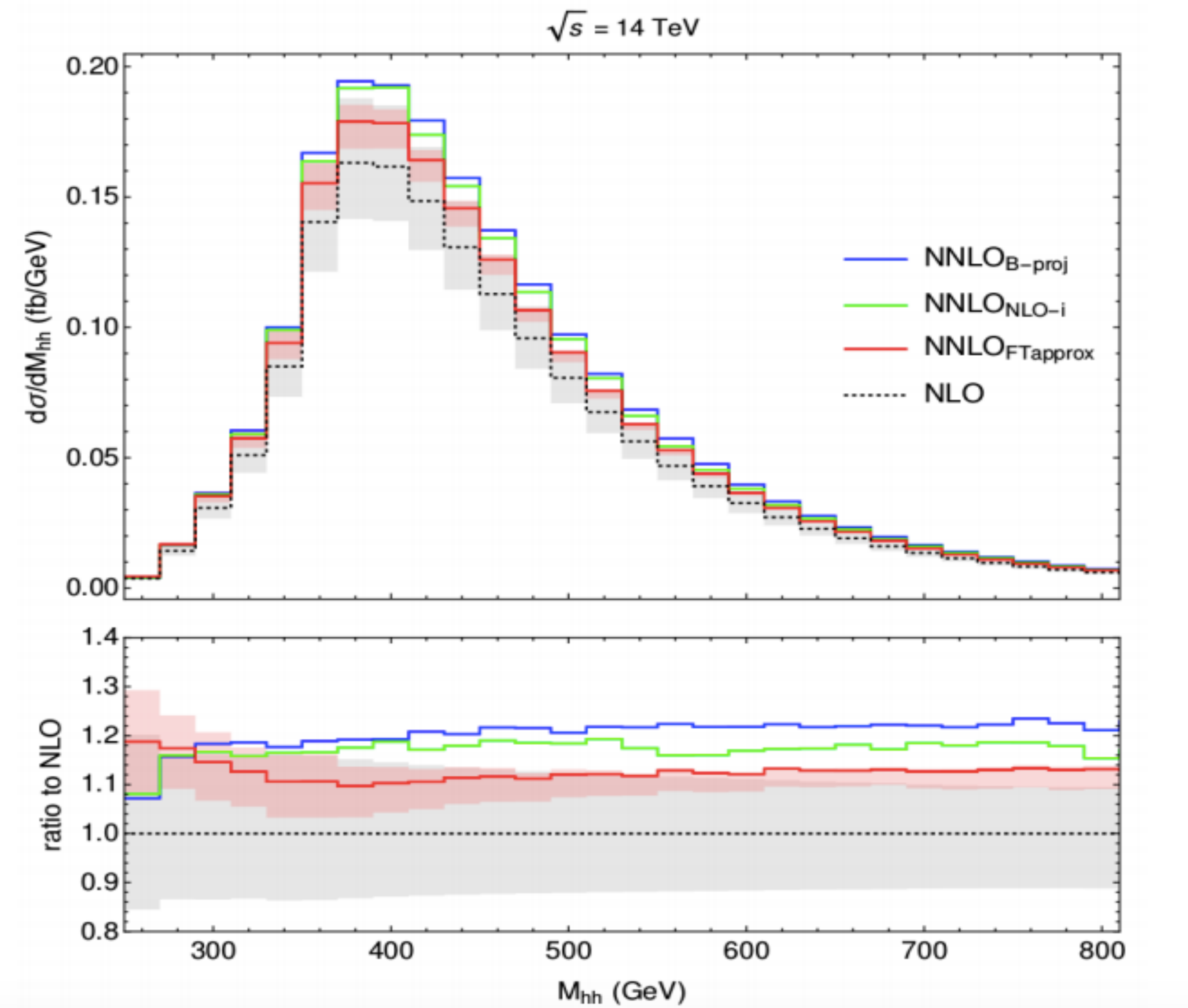


NNLO-large mt
J. Grigo, K. Melnikov, and M. Steinhauser;
D. de Florian and J. Mazzitelli,
Resummed:
D. Y. Shao, C. S. Li, H. T. Li, and J. Wang
D. de Florian and J. Mazzitelli,

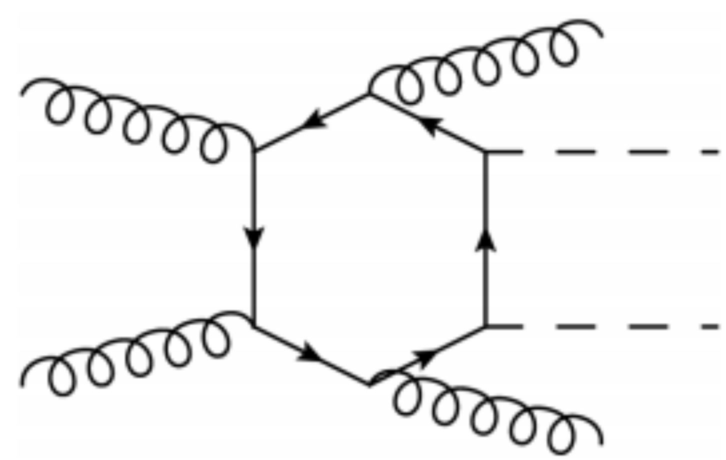
LO: Glover and van der Bij, Plen, Spira, Zerwas,
NLO: Dawson, Dittmair, Spira,
NLO-mt exp: J. Grigo, J. Hoff, K. Melnikov, and M. Steinhauser R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, P. Torrielli, E. Vryonidou, and M. Zaro; J. Grigo, K. Melnikov, and M. Steinhauser; F. Maltoni, E. Vryonidou, and M. Zaro; J. Grigo, J. Hoff, and M. SteinhauserG. Degrassi, P. P. Giardino, and R. Groeber
NLO-exact mt: S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, U. Schubert, and T. Zirke

At NNLO (approx) from gluon fusion

Grazzini, Heinrich, Jones, Kallweit, Kerner, Lindert,Javier Mazitelli



- Corrections of the order of a 12%
- Uncertainties are small
- Better convergence



N3LO from gluon fusion

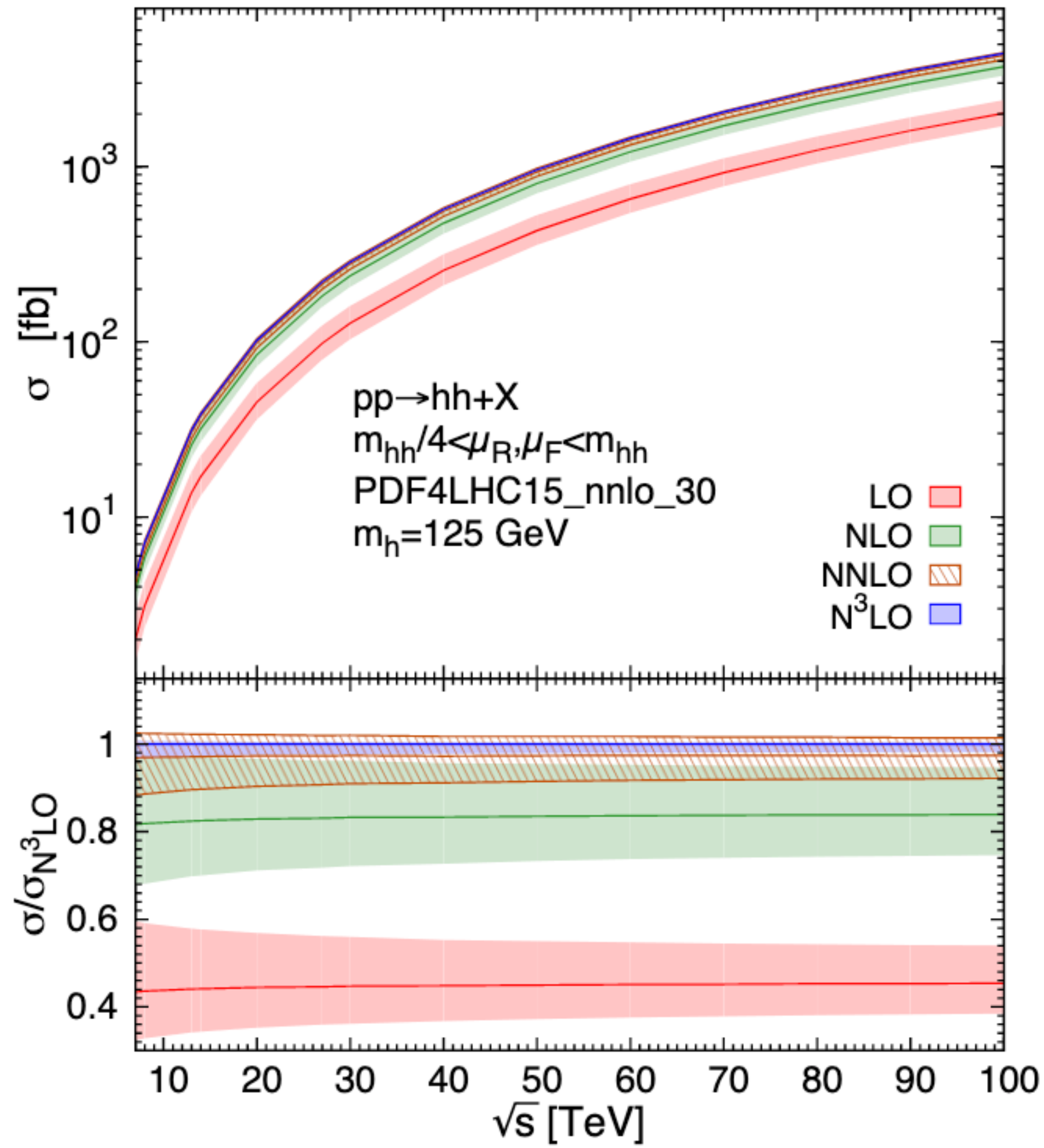


FIG. 3: The inclusive total cross sections for Higgs boson pair production at proton-proton colliders as a function of the collision energy. The bands represent the scale uncertainties. The bottom panel shows the ratios to the N^3LO result.

Long-Bin Chen, Hai Tao Li, Hua-Sheng Shao, Jian Wang

3% increase at N3LO

2 - 3% scale uncertainty

3.3 % PDF uncertainty

N3LO from gluon fusion

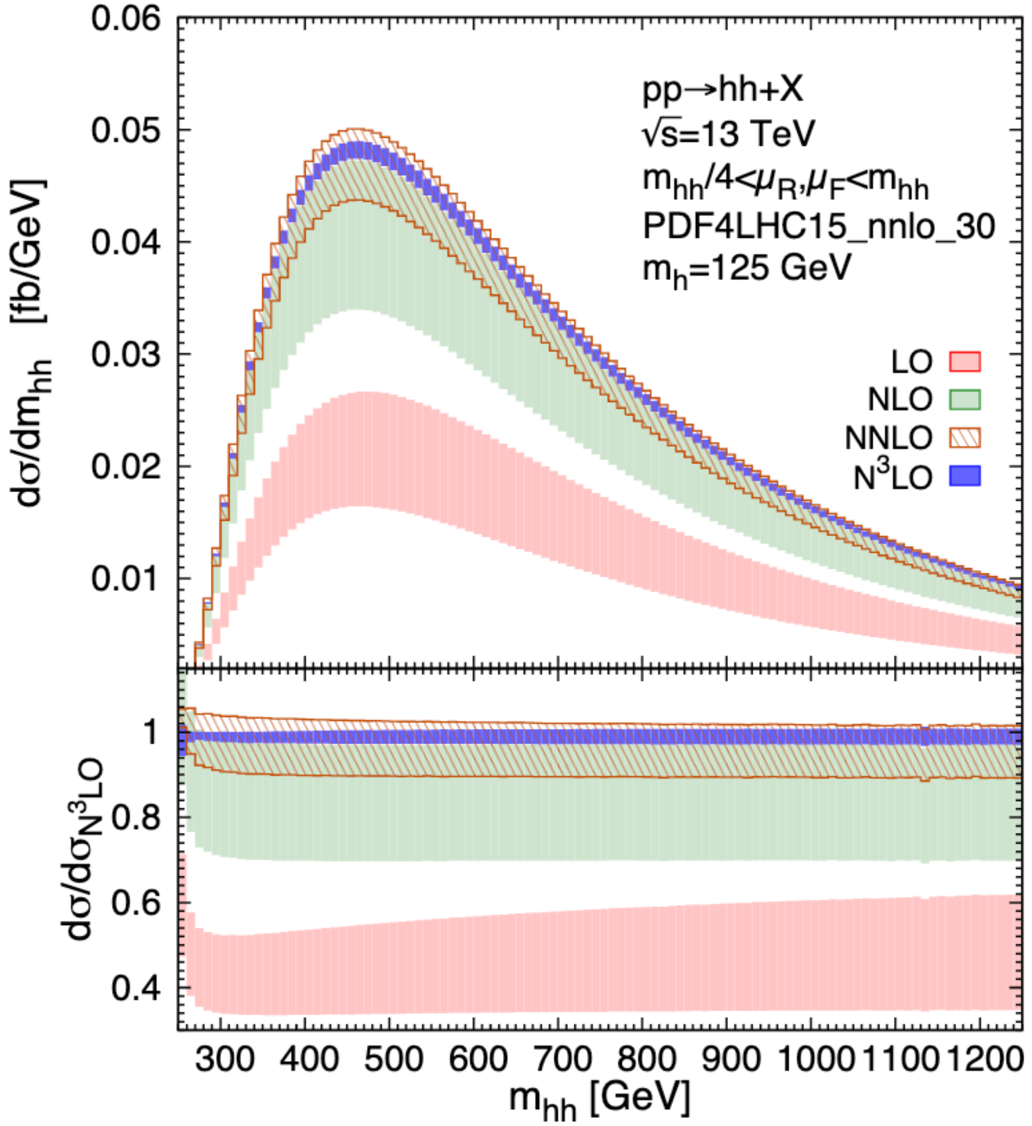


FIG. 5: Invariant mass distributions for Higgs boson pair production at the LHC with $\sqrt{s} = 13$ TeV. The bands represent the scale uncertainties. The red, green, brown and blue bands correspond to the LO, NLO, NNLO and N^3LO predictions, respectively. The bottom panel shows the ratios to the N^3LO distribution.

Long-Bin Chen, Hai Tao Li, Hua-Sheng Shao, Jian Wang

Infrared divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Quantum Field Theories with massless (even almost) particles encounter two kinds of infrared divergences:

Soft :

Soft divergences arise due to the presence of massless gauge bosons.

Collinear :

Collinear divergences arise when atleast two massless particles become collinear to each other

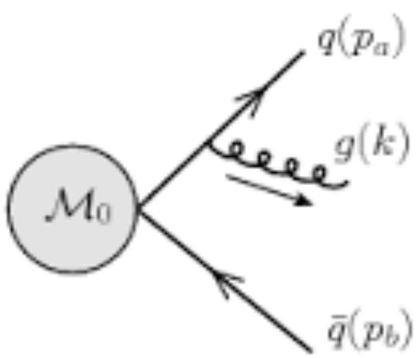
- Collinear gauge fields
- When the hard scale of the process is much larger than mass of the matter fields,

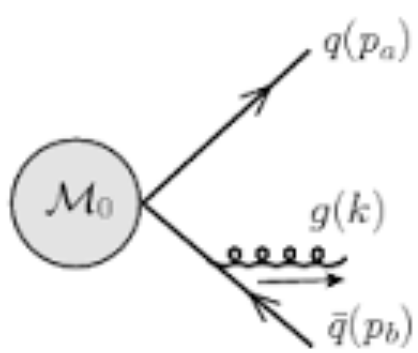
Infrared divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

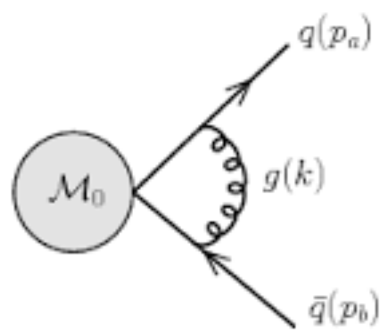
In the Limit $k \rightarrow p \quad (p_a \text{ or } \mathrm{p_b})$ $m_a, m_b << Q$

Real emission





Virtual



$$\frac{1}{(p+k)^2} = \frac{1}{2p^0 k^0 (1-\cos\theta)}$$

$k^0 \rightarrow 0$ Soft divergence

$\cos\theta \rightarrow 0$ Collinear divergence

Infrared divergences

[S Weinberg]

"In (Yang-Mills theory) a soft photon (gluon) *emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons* (gluons), *and so on, building up a cascade of soft massless particles each of which contributes an infra-red divergence.*

The elimination of such complicated interlocking infra-red divergences would certainly be a Herculean task, and might not even be possible."

S. Weinberg, Phys. Rev. 140B (1965)

Infrared Safety

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Physical processes that happen at Long distances are responsible for these divergences.

Long distance physics is associated to configurations that are experimentally indistinguishable

Measurable quantities are not sensitive to soft and Collinear divergences

[Bloch, Nordsieck, Kinoshita,: Lee, Nauenberg]

Bloch and Nordsieck Theorem

Soft Singularities cancel between real and virtual processes when one adds up all states which are indistinguishable by virtue of the energy resolution of the apparatus.

$$\exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}\exp\left\{+\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{\Delta E}{\lambda}\right)\right\} \longrightarrow \left(\frac{\Delta E}{E}\right)^{\alpha\mathcal{K}/\pi}$$

Kinoshita, Lee and Nauenberg Theorem

Both soft and collinear singularities cancel when the summation is carried out among all the mass degenerate states.

Catani's proposal

[Catani]

Upto Two loop !

$$\left[1 - \frac{\alpha_s}{2\pi} \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon)\right] |\mathcal{M}_n(\epsilon, \{p\})\rangle$$

Universal IR Subtraction Operators
depend only on
Process independent

Soft and Collinear

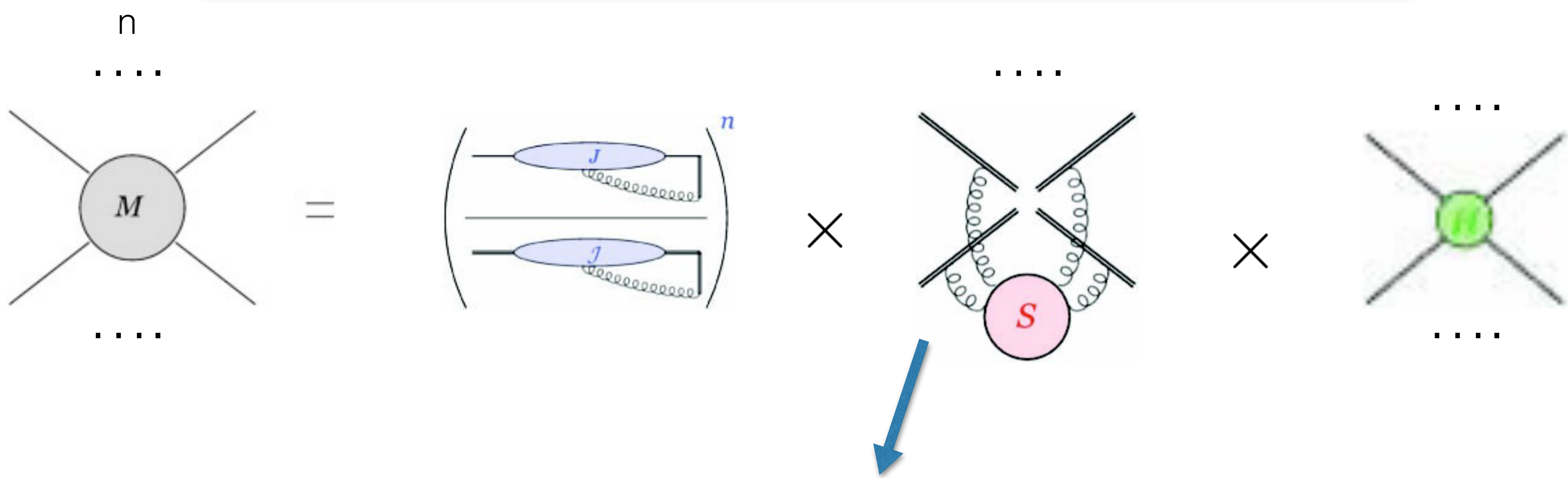
Anomalous Dimensions

Sterman's proof using factorisation

On-shell QCD amplitude in color basis:

[G. Sterman, M Tejeda-Yeomans]

$$\mathcal{M}^{[\mathrm{f}]}_{\{r_i\}}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \sum_{L=1}^{N^{[\mathrm{f}]}} \mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) (c_L)_{\{r_i\}}$$



$$\mathcal{M}^{[\mathrm{f}]}_{L}\left(\beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \prod_{i=1}^{n+2} J^{[i]}\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) S^{[\mathrm{f}]}_{LI}\left(\beta_j, \frac{Q'^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2), \epsilon\right) H^{[\mathrm{f}]}_{I}\left(\beta_j, \frac{Q^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2)\right)$$

Collinear Soft Hard

Soft+Virtual+Hard

DY/Higgs production cross section:

$$\sigma(q^2,\tau) = \sigma_0(\mu_R^2) \int \frac{dz}{z} \Phi_{ab}\left(\frac{\tau}{z}, \mu_F^2\right) \Delta_{ab}(q^2, \mu_F^2, z)$$

$$\Delta_{ab}(q^2,\mu_i^2,z) = \Delta_{ab}^{SV}(q^2,\mu_i^2,z) + \Delta_{ab}^{H}(q^2,\mu_i^2,z)$$

Soft + Virtual

Hard

Nielson PolyLogs

$$S_{np}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 dy \, \frac{\log^{n-1}(y)\log^p(1-zy)}{y}$$

$$\delta(1-z_i)$$

$$\left(\frac{\ln(1-z_i)}{(1-z_i)}\right)_+$$

- Next to SV

$$\Delta_{ab}^{NSV}(z) = \sum_{k=0}^{\infty} C_i^{NSV} \log^k(1-z)$$

- Next to next to....to soft

$$\Delta_{ab}^{N^n SV}(z) = \sum_{k=1}^{\infty} d_k (1-z)^k$$

Soft+Virtual

Soft plus Virtual

$$\Delta^{SV}_{ab}(z) = \delta_{ab}\Delta_{a,\delta}\delta(1-z) + \delta_{ab}\sum_{j=0}^{\infty}\Delta_{a,\mathcal{D}_j}\left(\frac{\log^j(1-z)}{1-z}\right)_+$$

Perturbatively Calculable

$$\Delta_{a,\delta} = \sum_k a_s^k \Delta^{(k)}_{a,\delta}$$

$$\Delta_{a,\mathcal{D}_j} = \sum_k a_s^k \Delta^{(k)}_{a,\mathcal{D}_j}$$

Sensitive to Soft and Collinear parsons

$$\frac{1}{(p+k)^2} = \frac{1}{2p^0k^0(1-\cos\theta)}$$

$k^0 \to 0$ Soft

$\cos\theta \to 1$ Collinear

- Divergences are controlled by Soft and Collinear Anomalous dimensions - ex: Cusp A, collinear B, soft-f etc
- Soft and Collinear regions are Universal
- RGE in IR sector allows for All Order Prediction

Mellin Moments and large N

Mellin Moment:

$$f_N = \int_0^1 dz \; z^{N-1} f(z)$$

Threshold limit $z \to 1$ in z-Space translates to $N \to \infty$ in N-Space

$N \to \infty$ Taking into account SV and NSV terms

$$\left(\frac{\log(1-z)}{1-z}\right)_+ = \frac{\log^2 N}{N} - \frac{\log N}{2N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\log^k(1-z) = \frac{\log^k N}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

N-Space structure

Mellin moment of CFs

Sterman, Catani, Trentedue

$$\Delta_N^c = \int_0^1 dz\ z^{N-1} \Delta_c(z)$$

In N-Space $N \to \infty$ We can predict tower of log N s

$$\Delta_N^c = 1 + a_s \Big[c_1^2 \log^2 N + c_1^1 \log N + c_1^0 + d_1^1 \frac{\log N}{N} + \mathcal{O}(1/N) \Big]$$

$$+ a_s^2 \Big[c_2^4 \log^4 N + \cdots + c_2^0 + d_2^3 \frac{\log^3 N}{N} + \cdots + \mathcal{O}(1/N) \Big]$$

$$+ \cdots$$

$$+ a_s^n \Big[c_n^{2n} \log^{2n} N + \cdots + d_n^{2n-1} \frac{\log^{2n-1} N}{N} + \cdots + \mathcal{O}(1/N) \Big]$$

$a_s \log N$ is of order \`one\` when a_s is very small

Spoils the trucation of the series

Sterman-Catani-Trentedue

Sterman (`87),Catani,Trentedue `89

SUMMATION OF LARGE CORRECTIONS TO SHORT-DISTANCE HADRONIC CROSS SECTIONS

George STERMAN*

Institute for Advanced Study, Princeton, NJ 08540, USA

RESUMMATION OF THE QCD PERTURBATIVE SERIES FOR HARD PROCESSES

S. CATANI

Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, Dipartimento di Fisica, Università di Firenze, I-50125 Florence, Italy

L. TRENTADUE

Dipartimento di Fisica, Università di Parma, INFN, Gruppo Collegato di Parma, I-43100 Parma, Italy

$$\Delta_N(Q^2) \underset{N\to\infty}{=} \exp\left(2\int_0^1 \mathrm{d}x\, \frac{x^{N-1}-1}{1-x} \int_{Q^2}^{Q^2(1-x)} \frac{\mathrm{d}k^2}{k^2} A\left(\alpha_s\left((1-x)Q^2\right)\right) + \frac{3}{2}\frac{C_F}{\pi}\int_0^1 \mathrm{d}x\, \frac{x^{N-1}-1}{1-x}\, \alpha_s\left((1-x)Q^2\right)\right) + \mathrm{O}\left(\alpha_s(\alpha_s \ln N)^n\right). \tag{3.25}$$

Resums threshold logarithms in large N

All order perturbative predictions

All order exponentiation can predict to all orders from lower orders:

$$\Delta_c(z) = \mathcal{C} \exp\left(\Psi^c\left(q^2, \mu_R^2, \mu_F^2, z, \varepsilon\right)\right)\Big|_{\varepsilon=0}$$

$$= \sum_{i=0}^{\infty} a_s^i \, \Delta_c^{(i)}(z)$$

$$\mathcal{D}_k = \left(\frac{\log^k(1-z)}{1-z}\right)_+$$

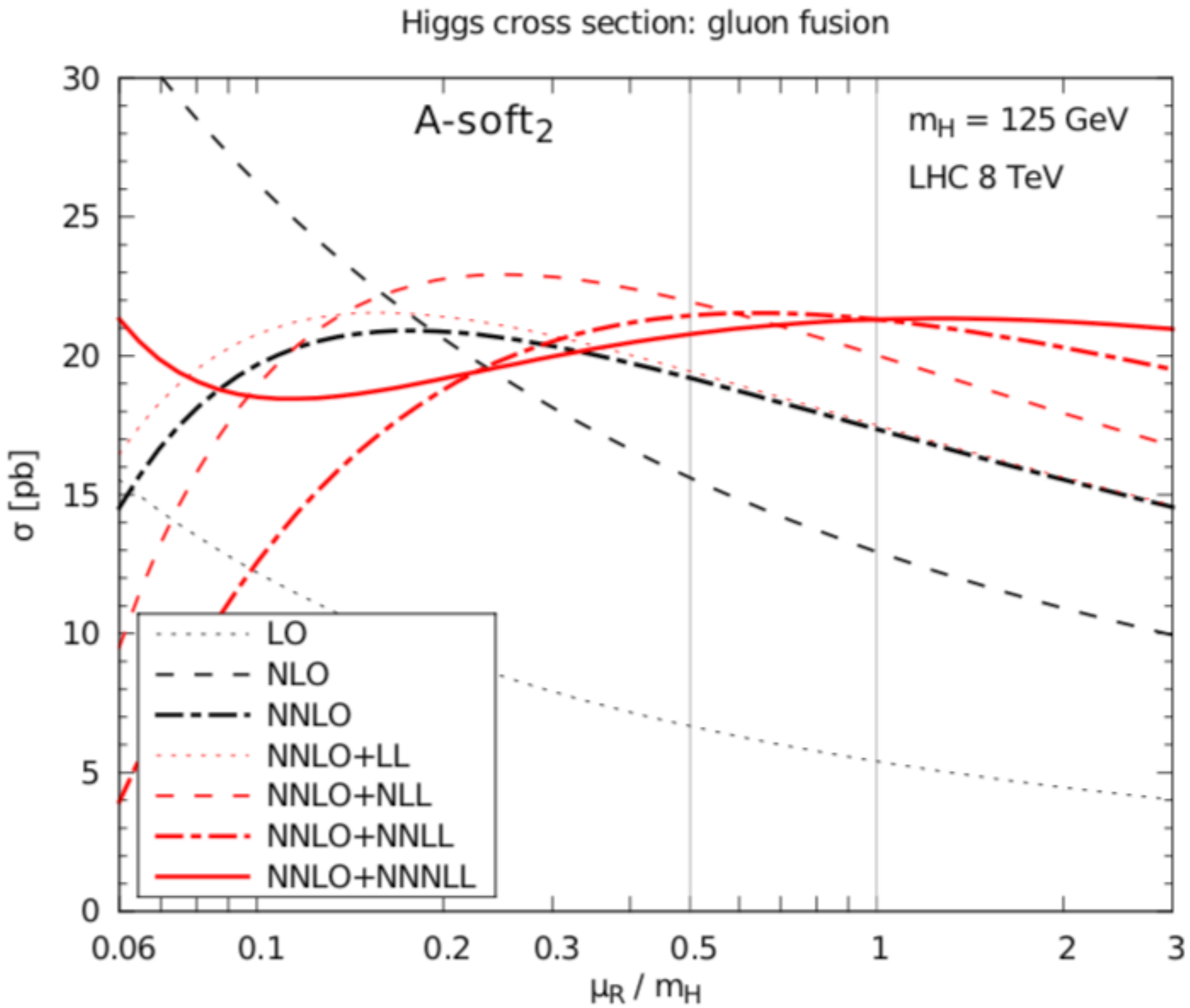
$$L_z = \log(1-z)$$

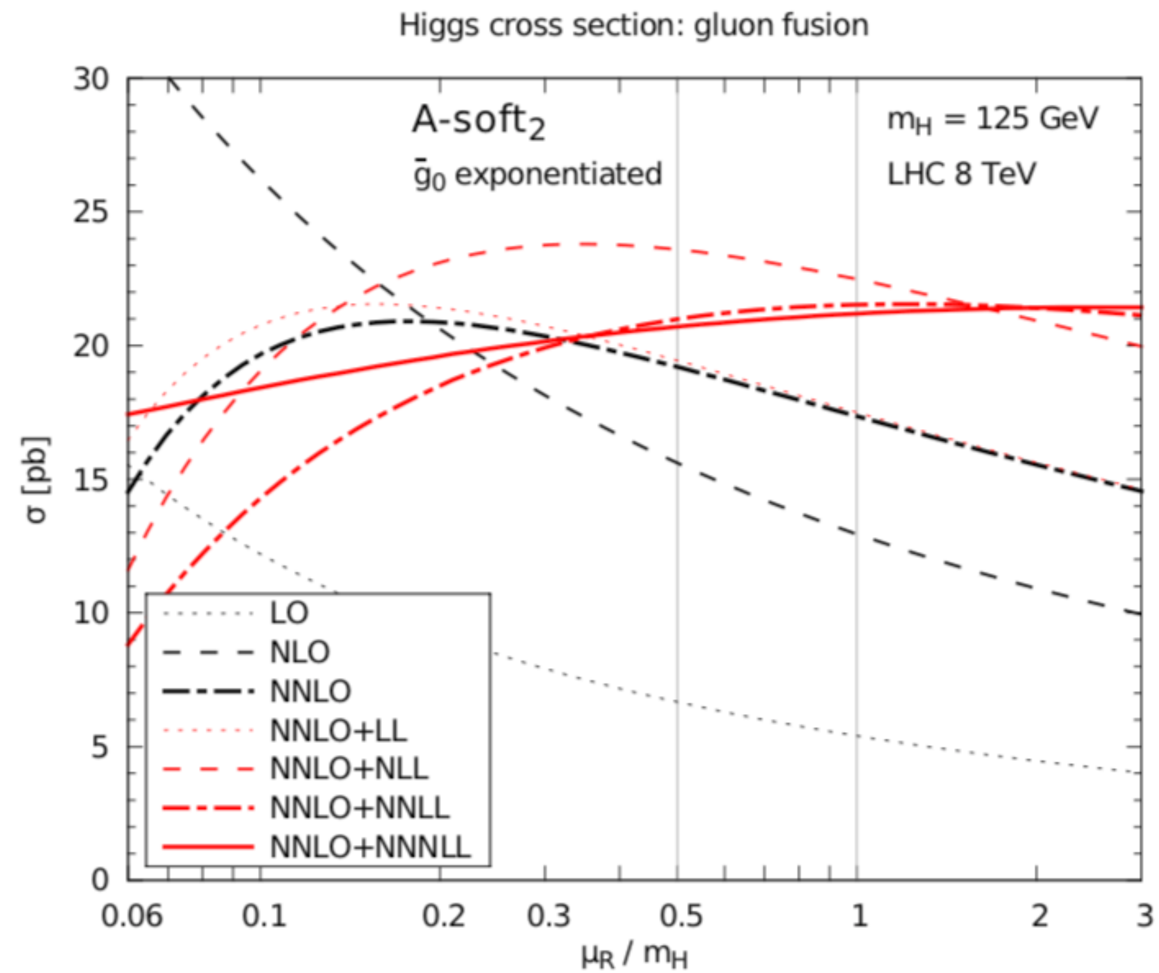
GIVEN				PREDICTIONS		
$\Psi_c^{(1)}$	$\Psi_c^{(2)}$	$\Psi_c^{(3)}$	$\Psi_c^{(n)}$	$\Delta_c^{(2)}$	$\Delta_c^{(3)}$	$\Delta_c^{(i)}$
$\mathcal{D}_0, \mathcal{D}_1, \delta$ L_z^1, L_z^0				$\mathcal{D}_3, \mathcal{D}_2$ L_z^3	$\mathcal{D}_5, \mathcal{D}_4$ L_z^5	$\mathcal{D}_{(2i-1)}, \mathcal{D}_{(2i-2)}$ $L_z^{(2i-1)}$
	$\mathcal{D}_0, \mathcal{D}_1, \delta$ L_z^2, L_z^1, L_z^0				$\mathcal{D}_3, \mathcal{D}_2$ L_z^4	$\mathcal{D}_{(2i-3)}, \mathcal{D}_{(2i-4)}$ $L_z^{(2i-2)}$
		$\mathcal{D}_0, \mathcal{D}_1, \delta$ $L_z^3, \cdots, L_z^0$				$\mathcal{D}_{(2i-5)}, \mathcal{D}_{(2i-6)}$ $L_z^{(2i-3)}$
			$\mathcal{D}_0, \mathcal{D}_1, \delta$ $L_z^n, \cdots, L_z^0$			$\mathcal{D}_{(2i-(2n-1))}, \mathcal{D}_{(2i-2n)}$ $L_z^{(2i-n)}$

Resummation in g g- H

Ajjath et al

Cross sections at LHC (13 TeV and above) with SV



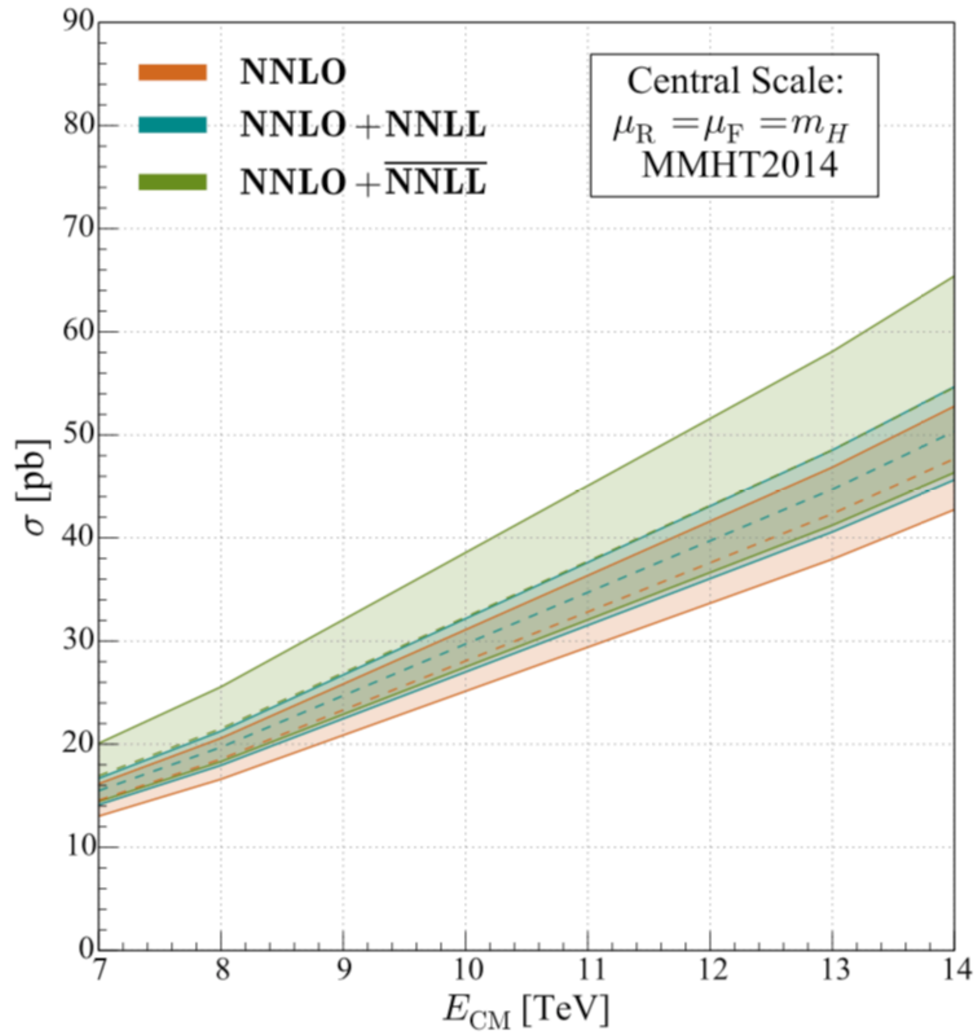


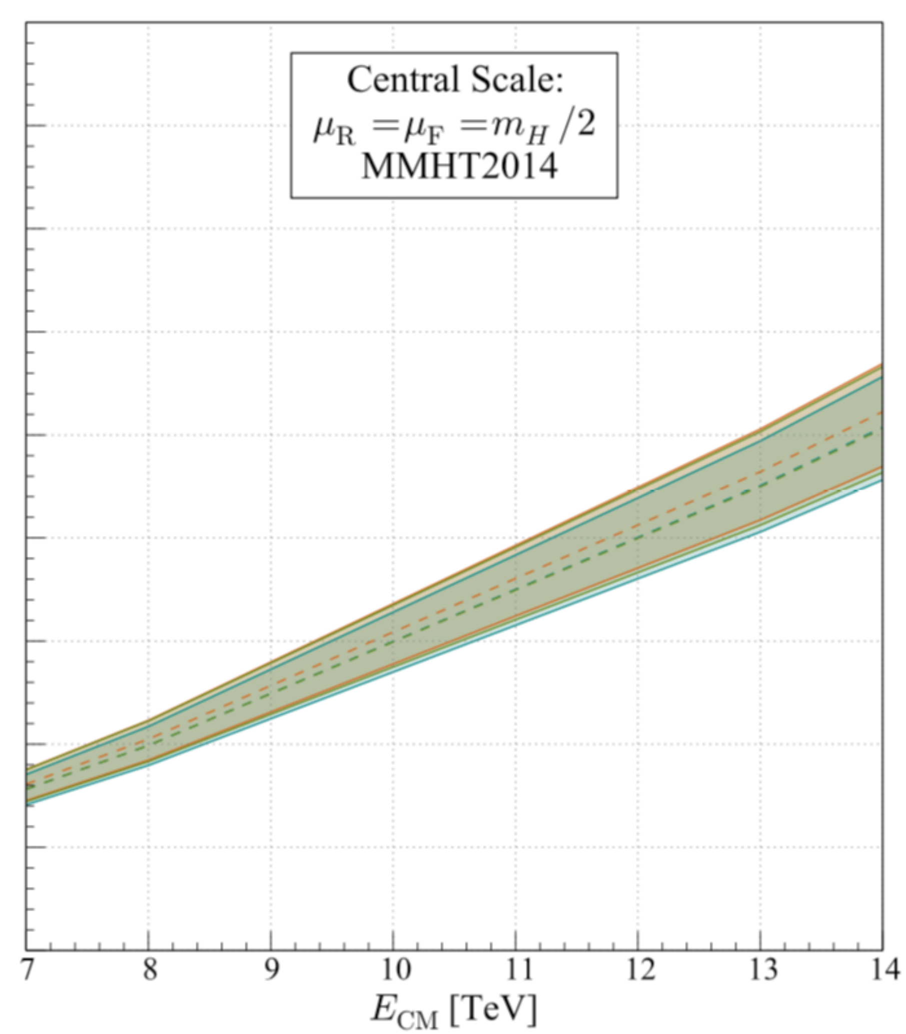
Resummation in g g- H

Ajjath et al

Cross sections at LHC (13 TeV and above) with SV and SV+ NSV

LO	LO + $\overline{\text{LL}}$	NLO	NLO + $\overline{\text{NLL}}$	NNLO	NNLO + $\overline{\text{NNLL}}$
$23.8940^{+0.80}_{-1.35}$	$42.7612^{+1.93}_{-2.96}$	$39.1681^{+0.93}_{-1.23}$	$41.0325^{+13.20}_{-7.64}$	$46.4304^{+0.44}_{-0.43}$	$44.9685^{+5.35}_{-3.40}$





Rapidity Distribution

Rapidity Distribution of any colorless particle:

$$\frac{d\sigma^I}{dy} = \hat{\sigma}^I_B \sum_{ab=q,\bar{q},g} \int_{x_1^0}^1 \frac{dz_1}{z_1} \int_{x_2^0}^1 \frac{dz_2}{z_2} \hat{\mathcal{H}}^I_{ab}\left(\frac{x_1^0}{z_1}, \frac{x_2^0}{z_2}, \mu^2\right) \hat{\Delta}^I_{d,ab}\left(z_1, z_2, q^2, \mu^2\right)$$

DY production of lepton pairs $\quad \sigma^I = \frac{d\sigma^q\left(\tau, q^2, y\right)}{dq^2}\,.$

Higgs through gluon (bottom anti-bottom), $\quad \sigma^I = \sigma^{g(b)}\left(\tau, q^2, y\right)\,.$

Rapidity: $\quad y = \frac{1}{2}\ln\left(\frac{p_2\cdot q}{p_1\cdot q}\right) = \ln\left(\frac{x_1^0}{x_2^0}\right)\,, \qquad \tau = x_1^0 x_2^0$

Partonic Scaling variables:

$$z_1 = \frac{x_1^0}{x_1}, \qquad z_1 = \frac{x_2^0}{x_2}$$

Soft and Virtual terms

$$\Delta_d^I = \delta(1-z_1)\delta(1-z_2) + a_s\Bigg\{ c_1^{(1)}\delta(1-z_1)\delta(1-z_2)$$

$$+c_2^{(1)}\left(\frac{\ln(1-z_1)}{1-z_1}\right)_+ + R^{(1)}(z_1,z_2) + z_1 \leftrightarrow z_2\Bigg\} + \mathcal{O}(a_s^2)$$

$$\Delta_d^I(z_1,z_2) = \Delta_d^{I,SV}(z_1.z_2) + \Delta_d^{I,hard}(z_1,z_2)$$

Virtual , Soft $\delta(1-z_i)$ $\left(\frac{\ln(1-z_i)}{(1-z_i)}\right)_+$

Soft Gluon Resummation

Double Mellin Transformation:

$$\tilde{\Delta}_d^{I,\mathrm{SV}}(\omega) = \int_0^1 dz_1\, z_1^{N_1-1} \int_0^1 dz_2\, z_2^{N_2-1}\, \Delta_d^{I,\mathrm{SV}}(z_1, z_2)$$

Resumed Rapidity distribution:

Ni dependent

$$\tilde{\Delta}_d^{\mathrm{SV},I}(\omega) = \tilde{g}_{d,0}^I(a_s) \exp\left(g_d^I(a_s,\omega)\right)$$

$$\omega = a_s \beta_0 \ln\left(\overline{N}_1 \overline{N}_2\right)$$

$$\overline{N}_i = e^{\gamma_E} N_i$$

Ni independent

All order resumed result

$$\tilde{\Delta}_{d,q}(N_1,N_2) = g^q_{d,0}(a_s)\exp\Bigg(\Bigg[\prod_{i=1,2}\int dz_i\; z_i^{N_i-1}\Bigg]$$

$$\Bigg[\delta(\overline{z}_2)\left(\frac{1}{\overline{z}_1}\left\{\int_{\mu_F^2}^{q^2\;\overline{z}_1}\frac{d\lambda^2}{\lambda^2}A^q\left(a_s(\lambda^2)\right)+D^q_d\left(a_s(q^2\;\overline{z}_1)\right)\right\}\right)_+$$

$$+\frac{1}{2}\left(\frac{1}{\overline{z}_1\overline{z}_2}\left\{A^q(a_s(z_{12}))+\frac{dD^q_d(a_s(z_{12}))}{d\ln z_{12}}\right\}\right)_+ + (\overline{z}_1\leftrightarrow\overline{z}_2)\Bigg]\Bigg)$$

Rapidity of Higgs at NNLO + NNLL

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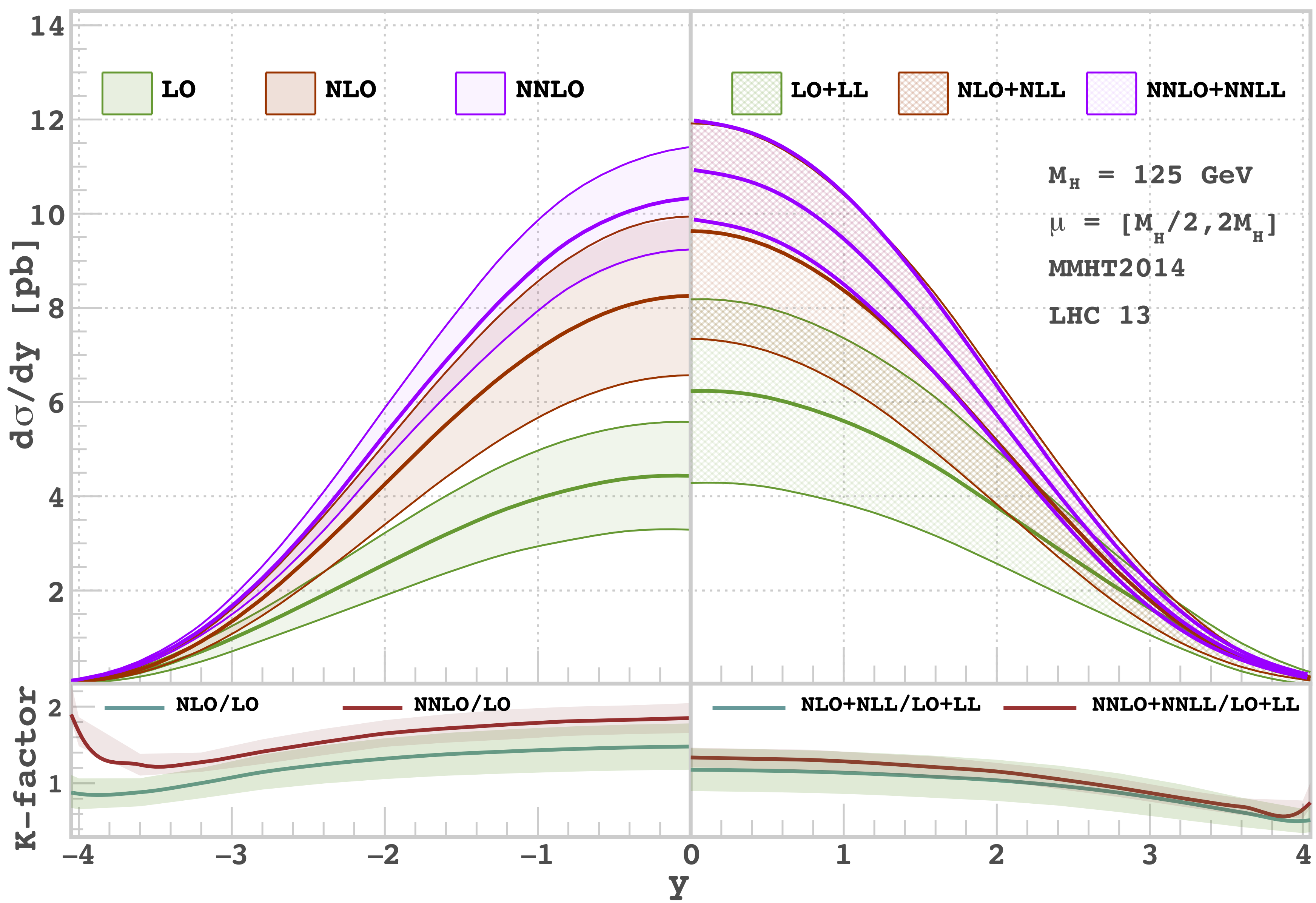

Fixed order CS   Resummed CS $\Longrightarrow$

At NNLO+NNLL result stabilises convergence of perturbation series !

$$M_H/2 \leq \mu_{R,F} \leq 2M_H$$

Rapidity of Higgs at NNLO + NNLL

NNLO+NNLL

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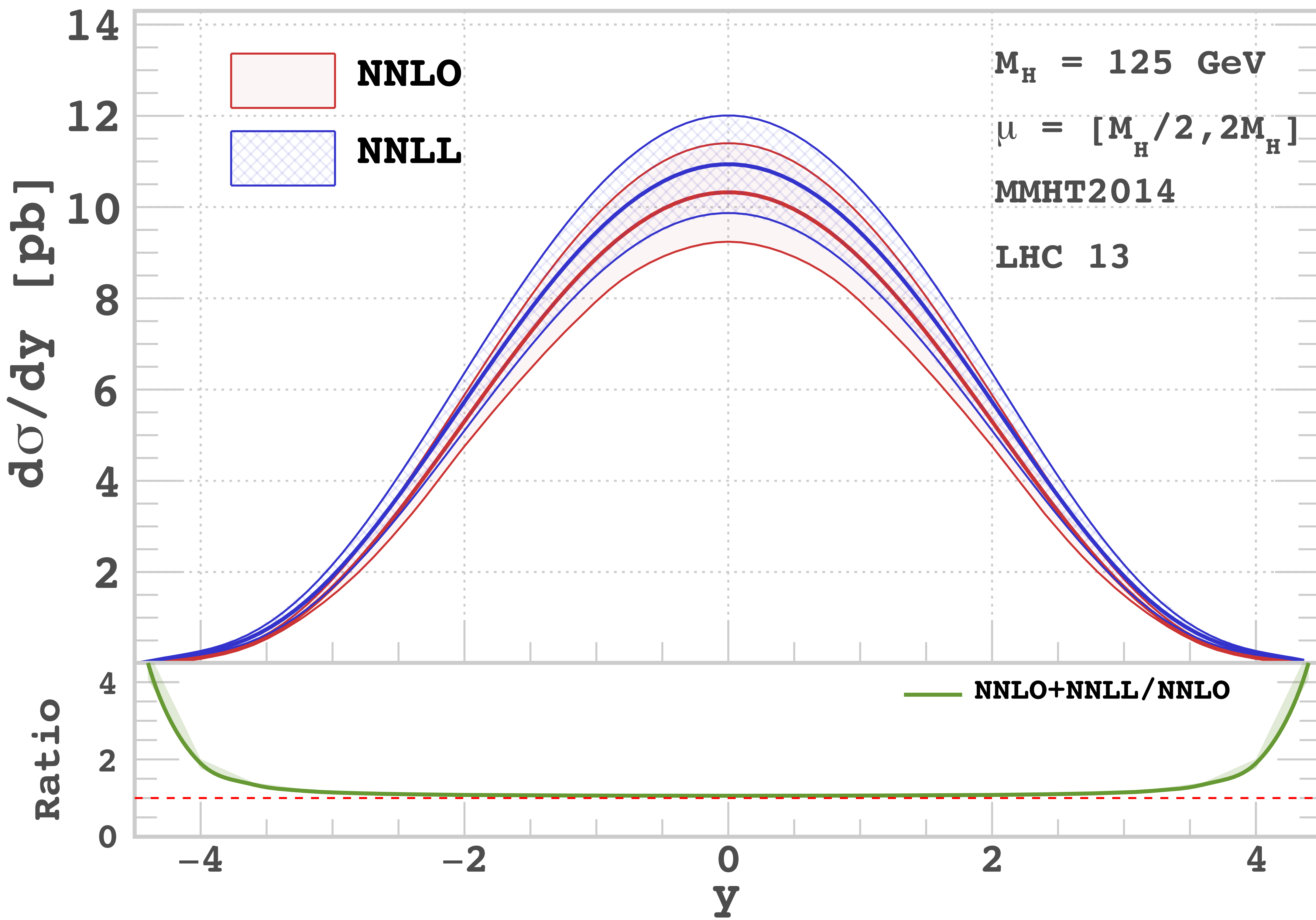

Rapidity of Higgs at NNLO + NNLL