Techniques for multi-loop computations

1. The Method of Differential Equations
2. Elliptic Integrals

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Section 1

Review

The Feynman integral

The Feynman integral for a Feynman graph G with n_{ext} external edges, n_{int} internal edges and l loops is given in D space-time dimensions by

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) = e^{l \varepsilon \gamma_{\mathrm{E}}} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i \pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}},$$

where each internal edge e_j of the graph is associated with a triple (q_j, m_j, ν_j),

$$q_j = \sum_{r=1}^{l} \lambda_{jr} k_r + \sum_{r=1}^{n_{\mathrm{ext}}-1} \sigma_{jr} p_r, \qquad \nu = \sum_{j=1}^{n_{\mathrm{int}}} \nu_j.$$

The coefficients λ_{jr} and σ_{jr} can be obtained from momentum conservation at each vertex of valency > 1.

Variables

The Feynman integral depends on:

- The **dimension of space-time** $D \in \mathbb{C}$
  (or more precisely on $D_{\mathrm{int}} \in \mathbb{N}$ and $\varepsilon \in \mathbb{C}$).
- The **exponents of the propagators** $(\nu_1, \dots, \nu_{n_{\mathrm{int}}})$.
  In principle we may allow $\nu_j \in \mathbb{C}$, but very often we will limit us to the case $\nu_j \in \mathbb{Z}$.
- **Kinematic variables**:
  - A scalar Feynman integral depends on the external momenta only through the Lorentz invariants $p_i \cdot p_j$.
  - A dimensionless Feynman integral depends on the Lorentz invariants, the internal masses and the scale μ only through the dimensionless ratios

$$\frac{-p_i \cdot p_j}{\mu^2}, \qquad \frac{m_i^2}{\mu^2}.$$

    We denote the dimensionless kinematic variables by $x_1, x_2, \dots$.

Kinematic variables

Notation:

number of independent kinematic variables:	N_B
independent kinematic variables:	$x_1, x_2, \dots, x_{N_B}$
Feynman integral:	$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B})$

Integration by parts

Integration-by-parts identities are based on the fact that within dimensional regularisation the **integral of a total derivative vanishes**

$$\int \frac{d^D k}{i\pi^{\frac{D}{2}}} \; \frac{\partial}{\partial k^\mu} \left[q^\mu \cdot f(k) \right] = 0,$$

i.e. there are no boundary terms.

Integration by parts

Integration-by-parts identities:
Within dimensional regularisation we have for any loop momentum k_i and any vector $q_{\mathrm{IBP}} \in \{p_1, \ldots, p_{N_{\mathrm{ext}}}, k_1, \ldots, k_l\}$

$$e^{l\varepsilon\gamma_{\mathrm{E}}} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \; \frac{\partial}{\partial \mathbf{k}_{\mathbf{i}}^{\mu}} \, q_{\mathrm{IBP}}^{\mu} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}} \quad = \quad 0.$$

Working out the derivatives leads to **relations among integrals** *with different sets of indices $(\nu_1, \ldots, \nu_{n_{\mathrm{int}}})$.*

Master integrals

Using

- integration-by-parts identities
- symmetries

we may express most of the integrals in terms of a few remaining integrals.
The remaining integrals are called **master integrals**.
We denote the indices of the master integrals by

$$\begin{aligned} \boldsymbol{\nu}_1 &= (\nu_{11}, \dots, \nu_{1 n_{\mathrm{int}}}), \\ \boldsymbol{\nu}_2 &= (\nu_{21}, \dots, \nu_{2 n_{\mathrm{int}}}), \\ & \dots \\ \boldsymbol{\nu}_{N_{\mathrm{master}}} &= (\nu_{N_{\mathrm{master}} 1}, \dots, \nu_{N_{\mathrm{master}} n_{\mathrm{int}}}). \end{aligned}$$

We define a N_{master}-dimensional vector $\vec{I}$ by

$$\vec{I} = \left(I_{\boldsymbol{\nu}_1}, I_{\boldsymbol{\nu}_2}, \dots, I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}} \right)^T.$$

Master integrals

Summary: *We may write any Feynman integral from a family of Feynman integrals as a linear combination of the master integrals*

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x_1, \dots, x_{N_B}) \quad = \quad \sum_{j=1}^{N_{\mathrm{master}}} c_j \, I_{\boldsymbol{\nu}_j}(D, x_1, \dots, x_{N_B}),$$

where the coefficients c_j are rational functions of D and the kinematic variables x.

Graph polynomials

Let G be a connected graph and $\mathcal{T}_1$ the set of its spanning trees.
The **first graph polynomial** *is given by*

$$\mathcal{U}(a) = \sum_{T \in \mathcal{T}_1} \prod_{e_i \notin T} a_i,$$

Let $\mathcal{T}_2$ be the set of its spanning 2-forests with respect to the internal edges. An element of $\mathcal{T}_2$ is denoted as (T_1, T_2). Let further denote P_{T_i} the set of external momenta of G attached to T_i. The **second graph polynomial** *is given by*

$$\mathcal{F}(a) = \mathcal{F}_0(a) + \mathcal{U}(a) \sum_{i=1}^{n_{\mathrm{int}}} a_i \frac{m_i^2}{\mu^2},$$

$$\mathcal{F}_0(a) = \sum_{(T_1,T_2) \in \mathcal{T}_2} \left(\prod_{e_i \notin (T_1,T_2)} a_i \right) \left(\sum_{p_j \in P_{T_1}} \sum_{p_k \in P_{T_2}} \frac{p_j \cdot p_k}{\mu^2} \right).$$

Dimensional-shift operators and raising operators

Dimensional-shift operators:

$$\mathbf{D}^{\pm} I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}\left(D, x_1, \dots, x_{N_B}\right) = I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}\left(D \pm 2, x_1, \dots, x_{N_B}\right)$$

Raising operators:

$$\mathbf{j}^{+} I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}\left(D, x_1, \dots, x_{N_B}\right) = \nu_j \cdot I_{\nu_1 \dots (\nu_j + 1) \dots \nu_{n_{\mathrm{int}}}}\left(D, x_1, \dots, x_{N_B}\right)$$

Note that we defined $\mathbf{j}^{+}$ such that it raises the index $\nu_j \rightarrow \nu_j + 1$ and multiplies the integral with a factor ν_j.

With this definition we have for example

$$\left(\mathbf{j}^{+}\right)^2 I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}\left(D, x_1, \dots, x_{N_B}\right) = \nu_j \left(\nu_j + 1\right) \cdot I_{\nu_1 \dots (\nu_j + 2) \dots \nu_{n_{\mathrm{int}}}}\left(D, x_1, \dots, x_{N_B}\right).$$

Dimensional shift relations

Recall

$$\mathbf{D}^{+} I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l\varepsilon\gamma_E}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{1}{\mathcal{U} \cdot \mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}},$$

$$\mathbf{j}^{+} I_{\nu_1 \dots \nu_j \dots \nu_{n_{\mathrm{int}}}}(D) = \frac{e^{l\varepsilon\gamma_E}}{\prod\limits_{k=1}^{n_{\mathrm{int}}} \Gamma(\nu_k)} \int\limits_{\alpha_k \geq 0} d^{n_{\mathrm{int}}}\alpha \left(\prod_{k=1}^{n_{\mathrm{int}}} \alpha_k^{\nu_k - 1} \right) \frac{\alpha_{\mathbf{j}}}{\mathcal{U}^{\frac{D}{2}}} e^{-\frac{\mathcal{F}}{\mathcal{U}}}.$$

Thus

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \mathcal{U}\left(\mathbf{1}^{+}, \dots, \mathbf{n}_{\mathrm{int}}{}^{+}\right) \mathbf{D}^{+} I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D).$$

Dimensional shift relations

Dimensional shift relations:

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D) = \mathcal{U}\left(\mathbf{1}^+, \dots, \mathbf{n}_{\mathrm{int}}{}^+\right) I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D+2).$$

- Let $\vec{I} = (I_{\boldsymbol{\nu}_1}, \dots, I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}})^T$ be a **basis in D space-time dimensions** and $\vec{I}' = (I'_{\boldsymbol{\nu}_1}, \dots, I'_{\boldsymbol{\nu}_{N_{\mathrm{master}}}})^T$ be a **basis in $(D+2)$ space-time dimensions**.
- Apply the shift relation to all integrals from $\vec{I}$ and reduce the integrals on the right-hand side with IBP-identities to $\vec{I}'$: We obtain a $(N_{\mathrm{master}} \times N_{\mathrm{master}})$-matrix S

$$\vec{I} = S \vec{I}'.$$

- Within dimensional regularisation the matrix S is invertible. Inverting this matrix allows us to express any master integral in $(D+2)$ dimensions as a linear combination of master integrals in D dimensions:

$$\vec{I}' = S^{-1} \vec{I}.$$

Section 2

Differential equations

The method of differential equations

Denote by $x = (x_1, ..., x_{N_B})$ the kinematic variables (scalar products of external momenta and internal masses squared).

We want to calculate

$$I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}(D, x)$$

1. *Find a differential equation with respect to the kinematic variables for the Feynman integral (always possible).*
2. *Transform the differential equation into a simple form (**bottle neck**).*
3. *Solve the latter differential equation with appropriate boundary conditions (always possible).*

Subsection 1

Deriving the differential equation

Differential equations

Let x_k be a kinematic variable. Let $I_i \in \{I_1, ..., I_{N_{\mathrm{master}}}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the **derivative as a linear combination of the master integrals**.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_F} a_{ij} I_j$$

Differential equations

The second Symanzik polynomial $\mathcal{F}$ is linear in the kinematic variables x_j. Set

$$\mathcal{F}'_{x_j}(a) = \frac{\partial}{\partial x_j}\mathcal{F}(a).$$

From the Schwinger parameter representation:

$$\frac{\partial}{\partial x_j} I_{\nu_1\dots\nu_{n_{\mathrm{int}}}}(D,x) = -\mathcal{F}'_{x_j}\left(\mathbf{1}^+,\dots,\mathbf{n}_{\mathrm{int}}{}^+\right) I_{\nu_1\dots\nu_{n_{\mathrm{int}}}}(D+2,x)$$

On the right-hand side:

- Reduce integrals to a basis in $(D+2)$ dimensions.
- Convert basis integrals from $(D+2)$ to D dimensions.

Differential equations

Let us formalise this:

$I = \left(I_1, ..., I_{N_{\mathrm{master}}}\right)$, set of master integrals,

$x = \left(x_1, ..., x_{N_B}\right)$, set of kinematic variables the master integrals depend on.

We obtain a **system of differential equations**

$$dI + AI \;=\; 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A \;=\; \sum_{i=1}^{N_B} A_i dx_i,$$

satisfying the integrability condition

$$dA + A \wedge A \;=\; 0.$$

Differential equations in ε-form

The system of differential equations is particular simple, if A is of the form

$$A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j,$$

where

- C_j is a $N_{\text{master}} \times N_{\text{master}}$-matrix, whose entries are (rational or integer) numbers,
- the **only dependence on ε** is **given by** the **explicit prefactor**,
- the differential one-forms ω_j have **only simple poles**.

Section 3

Solving a differential equation in ε-form

Solving a differential equation in ε-form

Assume

1. The differential equation for $\vec{I}$ is in **ε-form**:

$$(d + A)\vec{I} = 0, \qquad A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j.$$

2. All master integrals have a **Taylor expansion** in ε:

$$I_{\nu_i}(\varepsilon, x) = \sum_{j=0}^{\infty} I^{(j)}_{\nu_i}(x) \cdot \varepsilon^j.$$

3. We know suitable **boundary values** for all master integrals.

Solving a differential equation in ε-form

We plug the Taylor expansion into the differential equation

$$\left(d + \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k \right) \left(\sum_{j=0}^{\infty} \vec{I}^{(j)}(x) \cdot \varepsilon^j \right) = 0,$$

and compare term-by-term in the ε-expansion.

We obtain

$$\begin{aligned} d\vec{I}^{(0)}(x) &= 0, \\ d\vec{I}^{(j)}(x) &= - \sum_{k=1}^{N_L} \omega_k \, C_k \, \vec{I}^{(j-1)}(x), \quad j \geq 1. \end{aligned}$$

Chen's iterated integrals

Definition

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \;=\; \gamma^* \omega_j.$$

The **iterated integral** is defined by

$$I_\gamma(\omega_1,\dots,\omega_k;\lambda) \;=\; \int\limits_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \dots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Multiple polylogarithms

We are interested in differential one-forms, which have only simple poles.

The simplest case:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{d\lambda}{\lambda - z_j}.$$

Definition (Multiple polylogarithms)

$$G(z_1, ..., z_k; \lambda) = \int\limits_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int\limits_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} ... \int\limits_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}, \quad z_k \neq 0$$

The method of differential equations

Example

One integral I in one variable x with **boundary condition $I(0) = 1$**. Consider the differential equation

$$(d + A) I = 0, \qquad A = -\varepsilon \frac{dx}{x-1}.$$

Then

$$I(x) = 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + ...$$

Multiple polylogarithms

Definition based on **iterated integrals**:

$$G(z_1, ..., z_k; y) = \int\limits_0^y \frac{dt_1}{t_1 - z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

Definition based on **nested sums**:

$$\mathrm{Li}_{m_1, m_2, ..., m_k}(x_1, x_2, ..., x_k) = \sum_{n_1 > n_2 > ... > n_k > 0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot ... \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Conversion:

$$\mathrm{Li}_{m_1, ..., m_k}(x_1, ..., x_k) = (-1)^k G_{m_1, ..., m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, ..., \frac{1}{x_1 ... x_k}; 1\right)$$

Short hand notation:

$$G_{m_1, ..., m_k}(z_1, ..., z_k; y) = G(\underbrace{0, ..., 0}_{m_1 - 1}, z_1, ..., z_{k-1}, \underbrace{0 ..., 0}_{m_k - 1}, z_k; y)$$

Weights

Define the **weight** *of a multiple polylogarithm as*

$$\begin{aligned}\text{weight}\left(G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y)\right) &= m_1+\cdots+m_k,\\ \text{weight}\left(\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k)\right) &= m_1+\cdots+m_k.\end{aligned}$$

If the differential equation is in ε*-form, all* ω_j*'s are of the form*

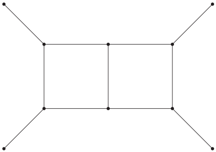
$$\omega_j = d\ln\left(p_j(x)\right),$$

where $p_j(x)$ *is a* **polynomial** *in the kinematic variables, and the boundary constants are of uniform weight, then the master integrals can be expressed in terms of multiple polylogarithms and are of uniform weight.*

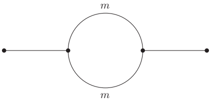
Section 4

Transformations of the differential equation

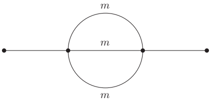
Examples



- Two-loop double box
  - 8 master integrals
  - 1 kinematic variable



- One-loop bubble
  - 2 master integrals
  - 1 kinematic variable



- Two-loop sunrise
  - 3 master integrals
  - 1 kinematic variable

Subsection 1

Fibre bundles

Fibre bundles

A fibre bundle consists of the following elements:

- A differentiable manifold E called the **total space**.
- A differentiable manifold M called the **base space**.
- A differentiable manifold F called the **fibre**.
- A **projection** $\pi : E \rightarrow M$. The inverse image $\pi^{-1}(p) = F_p$ is called the fibre at p.
- A Lie group G called the **structure group**, which acts on F from the left.

Principal bundles, vector bundles and connections

- A **principal bundle** P is a fibre bundle, whose fibre is identical with the structure group G.
- A **vector bundle** is a fibre bundle, whose fibre is a vector space. The dimension r of the fibre F is called the **rank** of the vector bundle.
- A **connection one-form** ω, which takes values in the Lie algebra $\mathfrak{g}$ of G, is a projection of $T_u P$ onto the vertical component $V_u P \cong \mathfrak{g}$, such that the horizontal subspaces $H_u P$ and $H_{ug} P$ on the same fibre are related by a linear map induced by $g \in G$.
- Denote by ***A*** the **pull-back** of ω by a section $s : M \to P$ to M:

$$A = s^* \omega.$$

A defines a **covariant derivative**:

$$\nabla = d + A.$$

Fibre bundles in physics

- **Quarks (QCD)**

Base space:	Minkowski space
Fibre:	3-dimensional vector space
Local connection one-form:	$A = \frac{g}{i} T^a A_\mu^a dx^\mu$

- **General relativity**

Base space:	(curved) space-time
Fibre:	Metric
Local connection one-form:	Levi-Civita connection

Feynman integrals

We have a vector bundle:

- Fibre spanned by the master integrals $I_{\nu_1}, ..., I_{\nu_{N_{\mathrm{master}}}}$.
  (The master integrals $I_{\nu_1}(x), \dots, I_{\nu_{N_{\mathrm{master}}}}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre.)
- Base space with coordinates $x = (x_1, ..., x_{N_B})$ corresponding to kinematic variables.
- Connection defined by the matrix A.

Transformations on this vector bundle:

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Transformations

- **Change the basis of the master integrals**

$$\vec{I}' = U \vec{I},$$

where $U(\varepsilon, x)$ is a $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix. The new connection matrix is

$$A' = U A U^{-1} + U d U^{-1}.$$

- **Perform a coordinate transformation on the base manifold**:

$$x_i' = f_i(x), \qquad 1 \leq i \leq N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Subsection 2

Fibre transformations

Overview

We seek a transformation $\vec{I}' = U\vec{I}$ such that $A' = UAU^{-1} + UdU^{-1}$ is simpler.

- **Block decomposition**
- Reduction to an univariate problem
- Picard-Fuchs operators
- Exploitung a master integral known to be of uniform weight
- Magnus expansion
- Moser's algorithm
- Leinartas decomposition
- **Maximal cuts and constant leading singularities**

Block decomposition

Order the set of master integrals $\vec{I} = \left(I_{\nu_1}, \dots, I_{\nu_{N_\text{master}}}\right)^T$ such that I_{ν_1} is the simplest integral and $I_{\nu_{N_\text{master}}}$ the most complicated integral.

The matrix A has a lower block-triangular structure:

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ A_3 & \multicolumn{2}{c}{A_2} & 0 \\ & & & 0 \\ A_6 & \multicolumn{2}{c}{A_5} & A_4 \end{pmatrix}$$

Diagonal blocks: A_1, A_2, A_4
Non-diagonal blocks: A_3, A_5, A_6

Diagonal blocks

Let's consider block A_2. We consider a transformation of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformed A' is given by

$$A' = \begin{pmatrix} A_1 & 0 & 0 \\ U_2 A_3 & \mathbf{U_2 A_2 U_2^{-1} + U_2 d U_2^{-1}} & 0 \\ A_6 & A_5 U_2^{-1} & A_4 \end{pmatrix}.$$

Suppose the block A_2 contains an unwanted term F and a remainder R:

$$A_2 = F + R.$$

The term F can be removed by a fibre transformation with U_2 given as a solution of the differential equation

$$dU_2^{-1} = -F U_2^{-1}.$$

Example

Assume that we have only one kinematic variable x_1 (e.g. $N_B = 1$) and that A_2 is of size (1×1) and given by

$$A_2 = \left(\frac{1}{x-1} + \frac{2\varepsilon}{x-1} \right) dx.$$

We would like to remove the first term $F = dx/(x-1)$ by a fibre transformation. We have to solve the differential equation

$$\frac{d}{dx} U_2^{-1} + \frac{1}{x-1} U_2^{-1} = 0.$$

A solution is easily found and given by

$$U_2^{-1} = \frac{C}{x-1}, \qquad U_2 = C^{-1} \left(x-1\right).$$

We may set $C = 1$ and $U_2 = x - 1$ is the sought-after transformation.

Non-diagonal blocks

Let us now consider block A_3. At this stage we would like to preserve the blocks A_1 and A_2. We consider a transformation of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ U_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -U_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformed A' is given by

$$A' = \begin{pmatrix} A_1 & 0 & 0 \\ \mathbf{A_3 - A_2 U_3 + U_3 A_1 - d U_3} & A_2 & 0 \\ A_6 - A_5 U_3 & A_5 & A_4 \end{pmatrix}.$$

Suppose the block A_3 contains an unwanted term F and a remainder R:

$$A_3 = F + R.$$

The term F can be removed by a fibre transformation with U_3 given as a solution of the differential equation

$$d U_3 + A_2 U_3 - U_3 A_1 = F.$$

Example

We again consider the case of one kinematic variable x (e.g. $N_B = 1$). We further assume that A_1 and A_2 are both blocks of size (1×1). Then A_3 is also a block of size (1×1). Assume that A_1 and A_2 are already in ε-form an given by

$$A_1 = \frac{\varepsilon dx}{x-1}, \qquad A_2 = \frac{2\varepsilon dx}{x-1}.$$

Assume further that F is given by

$$F = \frac{\mathbf{dx}}{(\mathbf{x}-\mathbf{1})^{\mathbf{2}}}.$$

We have to solve the differential equation

$$\left[\frac{d}{dx} + \frac{\varepsilon}{x-1}\right] U_3 = \frac{1}{(x-1)^2}.$$

A solution is given by

$$U_3 = \frac{1}{(1-\varepsilon)(1-x)}.$$

Subsection 3

Maximal cuts and constant leading singularities

Heuristic methods

- Suppose somebody gives us a transformation matrix U

$$\vec{I}' = U\vec{I}.$$

- It is easy to check if this fibre transformation transforms the differential equation to an ε-form. We simply calculate

$$A' = UAU^{-1} + UdU^{-1}$$

  and check if A' is in ε-form.
- This is a situation where a heuristic method may work well: Guessing a suitable U may outperform any systematic algorithm to construct the matrix U.

Feynman integrals with cuts

Recal: Baikov representation

$$I_{\nu_1 \dots \nu_n}(D, x_1, \dots, x_{N_B}) = C \int\limits_{\mathcal{C}} d^{N_V} z \; [\mathcal{B}(z)]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}$$

with integration contour $\mathcal{C}$.

Consider a **modified integration contour** $\mathcal{C}'$ such that

1. Integration-by-parts identities still hold.
2. The variation of the integral with respect to the kinematic variables comes entirely from the integrand.
3. The symmetries among the integrals are respected.

The maximal cut

Definition (Feynman integral with the internal edge e_j cut)

Baikov integral with a modified integration domain $\mathcal{C}'$:

- a small anti-clockwise circle around $z_j = 0$ in the complex z_j-plane,
- in all other variables the intersection of the original integration domain $\mathcal{C}$ with the hyperplane $z_j = 0$.

We may iterate the procedure and take multiple cuts. Of particular importance is the maximal cut:

Definition (Maximal cut)

Take for a Feynman integral $I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}$ the cut for all edges e_j for which $\nu_j > 0$.

Example

One-loop two-point function with equal internal masses:
Baikov polynomial ($x = -p^2/m^2$ and $\mu^2 = m^2 = 1$):

$$\mathcal{B}(z_1, z_2) = -\frac{1}{4}\left[(z_1 - z_2)^2 - 2x(z_1 + z_2) + x(4+x)\right],$$

Baikov representation of I_{11}:

$$I_{11} = \frac{e^{\varepsilon\gamma_{\mathrm{E}}} x^{-\frac{D-2}{2}}}{2\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)} \int\limits_{\mathcal{C}} d^2z \, \left[\mathcal{B}(z_1, z_2)\right]^{\frac{D-3}{2}} \frac{1}{z_1 z_2}.$$

Maximal cut:

$$\mathrm{MaxCut}\, I_{11} = (2\pi i)^2 \frac{e^{\varepsilon\gamma_{\mathrm{E}}} x^{-\frac{D-2}{2}}}{2\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)} \left(-\frac{1}{4} x(4+x)\right)^{\frac{D-3}{2}}.$$

In $D = 2 - 2\varepsilon$ dimensions we have to leading order in the ε-expansion:

$$\mathrm{MaxCut}\, I_{11}(2-2\varepsilon) = -\frac{4\pi}{\sqrt{-x(4+x)}} + \mathcal{O}(\varepsilon).$$

Constant leading singularities

- Denote the **integrands** of the master integrals by $\varphi_1, \dots, \varphi_{N_\mathrm{master}}$.
- Choose N_master **independent integration domains** $\mathcal{C}_1, \dots, \mathcal{C}_{N_\mathrm{master}}$. The integration domains are independent, if the $N_\mathrm{master} \times N_\mathrm{master}$-matrix with entries

$$\langle \varphi_i | \mathcal{C}_j \rangle = \int\limits_{\mathcal{C}_j} \varphi_i$$

has full rank.
- We are interested in choosing the integration domains $\mathcal{C}_j$ as simple as possible. Particular simple integration domains are products of circles around singular points. These correspond to residue calculations.

Constant leading singularities

- Let φ be the integrand of a Feynman integral I.
- Define d_{min} by

$$d_{\mathrm{min}} = \min_j \left(\mathrm{ldegree}\left(\langle \varphi | \mathcal{C}_j \rangle , \varepsilon \right) \right),$$

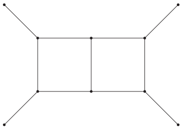
- We say that the Feynman integral I has **constant leading singularities**, if for all j

$$\mathrm{coeff}\left(\langle \varphi | \mathcal{C}_j \rangle , \varepsilon^{d_{\mathrm{min}}} \right) = \text{constant of weight zero},$$

- Integrals with constant leading singularities are a guess for a basis of master integrals, which puts the differential equation into an ε-form.

Example

- Consider the two-loop double box integral with vanishing internal masses, $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$ and $x = s/t$.



- This is a system with eight master integrals.
- Suppose we already found suitable master integrals, which puts the sub-system of the first six master integrals into an ε-form.
- Thus we are left with finding a fibre transformation, which transforms the **last sector**, consisting of the two master integrals $I_{111111100}$ and $I_{1111111(-1)0}$ into an ε-form.

Example

Consider the maximal cut of this sector for the integrals $I_{1111111\nu 0}$.
With $\mu^2 = t$ we have

$$\text{MaxCut } I_{1111111\nu 0} = (2\pi i)^7 \frac{2^{4\varepsilon}(s+t)^{\varepsilon}\, t^{3+\nu+3\varepsilon}}{4\pi^3\left(\Gamma\left(\frac{1}{2}-\varepsilon\right)\right)^2 s^{2+2\varepsilon}} \int\limits_{\mathcal{C}_{\text{MaxCut}}} dz_8\, z_8^{-1-2\varepsilon}\,(t-z_8)^{-1-\varepsilon}\,(s+t-z_8)^{\varepsilon}\, z_8^{-\nu}.$$

We now choose two independent integration domains:

$$\begin{aligned} \mathcal{C}_1 &: \text{ small circle around } z_8 = 0 \text{ for the } z_8\text{-integration},\\ \mathcal{C}_2 &: \text{ small circle around } z_8 = t \text{ for the } z_8\text{-integration}. \end{aligned}$$

We set

$$\varphi_{\nu} = \frac{2^{4\varepsilon}(s+t)^{\varepsilon}\, t^{3+\nu+3\varepsilon}}{4\pi^3\left(\Gamma\left(\frac{1}{2}-\varepsilon\right)\right)^2 s^{2+2\varepsilon}}\, z_8^{-1-2\varepsilon}\,(t-z_8)^{-1-\varepsilon}\,(s+t-z_8)^{\varepsilon}\, z_8^{-\nu} d^8 z.$$

Example

With $x = s/t$ we have

$$\langle \varphi_0 | \mathcal{C}_1 \rangle = \frac{64\pi^4}{x^2} + \mathcal{O}(\varepsilon), \qquad \langle \varphi_0 | \mathcal{C}_2 \rangle = -\frac{64\pi^4}{x^2} + \mathcal{O}(\varepsilon).$$

The integral

$$\text{MaxCut } I_{111111100} = \langle \varphi_0 | \mathcal{C}_{\text{MaxCut}} \rangle$$

does not have constant leading singularities, but it is **easy to fix** this issue:

- We multiply the integrand by x^2.
- If in addition we multiply by ε^4, the leading singularities are constants of weight zero.
- Strictly speaking we can only infer from the first term of the ε-expansion of $\langle \varphi_0 | \mathcal{C}_j \rangle$ that we should multiply by an ε-dependent prefactor, whose ε-expansion starts at ε^4. In this example we can verify a posteriori that ε^4 is the correct ε-dependent prefactor.

Example

Set

$$\varphi_0' = \varepsilon^4 x^2 \varphi_0 .$$

Then

$$\left\langle \varphi_0' \middle| \mathcal{C}_1 \right\rangle = 64\pi^4\varepsilon^4 + \mathcal{O}(\varepsilon), \qquad \left\langle \varphi_0' \middle| \mathcal{C}_2 \right\rangle = -64\pi^4\varepsilon^4 + \mathcal{O}(\varepsilon).$$

Thus

$$\mathrm{MaxCut}\left(\varepsilon^4 x^2 I_{111111100}\right) = \left\langle \varphi_0' \middle| \mathcal{C}_{\mathrm{MaxCut}} \right\rangle$$

has constant leading singularities.

Example

As this sector has two master integrals, we **need a second master integral**. We consider φ_{-1} and compute the leading singularities. We obtain

$$\langle \varphi_{-1} | \mathcal{C}_1 \rangle = 0 + \mathcal{O}(\varepsilon), \qquad \langle \varphi_{-1} | \mathcal{C}_2 \rangle = -\frac{64\pi^4}{x^2} + \mathcal{O}(\varepsilon).$$

It follows that

$$\mathrm{MaxCut}\left(\varepsilon^4 x^2 I_{1111111(-1)0}\right) = \left\langle \varepsilon^4 x^2 \varphi_{-1} | \mathcal{C}_{\mathrm{MaxCut}} \right\rangle$$

has constant leading singularities.

Example

It is easily verified, that the two master integrals

$$\varepsilon^4 x^2 I_{111111100} \quad \text{and} \quad \varepsilon^4 x^2 I_{1111111(-1)0}$$

put the **2×2-diagonal block** for this sector into an **ε-form**.

It remains to treat the off-diagonal block with entries $A_{i,j}$, $i \in \{7,8\}$, $j \in \{1,2,3,4,5,6\}$. This is most easily done with the methods discussed in the context of block decomposition. One finds

$$\begin{aligned} I'_{\boldsymbol{\nu}_7} &= \varepsilon^4 x^2 I_{111111100}, \\ I'_{\boldsymbol{\nu}_8} &= \varepsilon^4 x^2 I_{1111111(-1)0} + x \left[I'_{\boldsymbol{\nu}_6} + \frac{1}{2} \left(I'_{\boldsymbol{\nu}_5} + I'_{\boldsymbol{\nu}_4} - I'_{\boldsymbol{\nu}_2} - I'_{\boldsymbol{\nu}_1} \right) \right]. \end{aligned}$$

Lecture 2

Transformations

- **Change the basis of the master integrals**

$$\vec{I}' = U \vec{I},$$

where $U(\varepsilon, x)$ is a $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix. The new connection matrix is

$$A' = U A U^{-1} + U d U^{-1}.$$

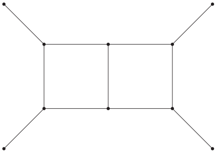
- **Perform a coordinate transformation on the base manifold**:

$$x'_i = f_i(x), \qquad 1 \leq i \leq N_B.$$

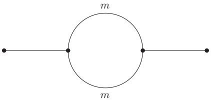
The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x'_j} dx'_j.$$

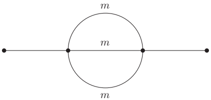
Examples



- Two-loop double box
  - 8 master integrals
  - 1 kinematic variable



- One-loop bubble
  - 2 master integrals
  - 1 kinematic variable



- Two-loop sunrise
  - 3 master integrals
  - 1 kinematic variable

Coordinate transformation on the base manifold

- The transformation to an ε-factorised form may introduce algebraic or transcendental functions.
- A coordinate transformation may lead to a nicer form.
  Examples:
  - **Square roots**:

$$x = \frac{(1-x')^2}{x'}, \quad x' = \frac{1}{2}\left(2+x-\sqrt{x(4+x)}\right) \quad \Rightarrow \quad \frac{dx}{\sqrt{x(4+x)}} = -\frac{dx'}{x'}$$

  - **Elliptic case**:

$$x = -9\frac{\eta(\tau)^4\eta(6\tau)^8}{\eta(3\tau)^4\eta(2\tau)^8}, \quad \tau = \frac{\psi_2(x)}{\psi_1(x)} \quad \Rightarrow \quad \left(\frac{\pi}{\psi_1(x)}\right)^2 \frac{12dx}{x(x+1)(x+9)} = 2\pi i\, d\tau$$

Subsection 4

Base transformations

Base transformations

Coordinate transformation on the base manifold:

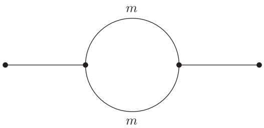
$$x'_i = f_i(x), \qquad 1 \le i \le N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x'_j} dx'_j.$$

Example

The one-loop two point function:



Master integrals:

$$\vec{I} = \begin{pmatrix} I_{10} \\ I_{11} \end{pmatrix}$$

Differential equation:

$$(d+A)\vec{I} = 0, \qquad A = \begin{pmatrix} 0 & 0 \\ \frac{1-\varepsilon}{2x} - \frac{1-\varepsilon}{2(x+4)} & \frac{1}{2x} - \frac{1-2\varepsilon}{2(x+4)} \end{pmatrix} dx.$$

Example

There is no fibre transformation **rational** in x and ε, which factors out ε. However, if we allow the transformation to be **algebraic**, we may achieve this goal.

$$\vec{I}' = U \vec{I}, \qquad U = \begin{pmatrix} 2\varepsilon(1-\varepsilon) & 0 \\ 2\varepsilon(1-\varepsilon)\sqrt{\frac{x}{4+x}} & 2\varepsilon(1-2\varepsilon)\sqrt{\frac{x}{4+x}} \end{pmatrix}.$$

For the transformed system we find

$$\left(d + A'\right)\vec{I}' = 0, \qquad A' = \varepsilon \begin{pmatrix} 0 & 0 \\ -\frac{dx}{\sqrt{x(4+x)}} & \frac{dx}{4+x} \end{pmatrix}.$$

Example

We have achieved that ε only appears as a prefactor, however we introduced non-rational functions: The differential one-form

$$\frac{dx}{\sqrt{x(4+x)}}$$

has square root singularities at $x = 0$ and $x = -4$.

Remark:

$$\frac{dx}{\sqrt{x(4+x)}} = d \ln \left(2 + x + \sqrt{x(4+x)} \right).$$

We see that in this case the argument of the logarithm is no longer a polynomial, but an **algebraic function** of x.

Example

Let's **define** x' by

$$x = \frac{(1-x')^2}{x'}.$$

The inverse relation reads

$$x' = \frac{1}{2}\left(2+x-\sqrt{x(4+x)}\right),$$

where we made a choice for the sign of the square root. We have

$$\frac{\partial x}{\partial x'} = -\frac{(1-x')^2}{x'^2}$$

and

$$\frac{dx}{\sqrt{x(4+x)}} = -\frac{dx'}{x'}, \qquad \frac{dx}{4+x} = \frac{2dx'}{x'+1} - \frac{dx'}{x'}.$$

Example

Thus in term of the new variable x' we have

$$\left(d + A'\right)\vec{I}' = 0, \qquad A' = \varepsilon \begin{pmatrix} 0 & 0 \\ \frac{dx'}{x'} & \frac{2dx'}{x'+1} - \frac{dx'}{x'} \end{pmatrix}.$$

The differential equation is now in **ε-form**:

- The dimensional regularisation parameter occurs only as a prefactor
- The only singularities of A' are simple poles.
- For the case at hand, A' has simple poles at $x' = 0$ and $x' = -1$.

Rationalising square roots

Consider

$$\sqrt{f(x_1, \dots, x_n)} \quad \text{and} \quad V(f) = \{x \in \mathbb{C}^n | f(x) = 0\}.$$

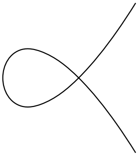
A point $p \in V$ is said to be of **multiplicity** $o \in \mathbb{N}$ if all partial derivatives of order $< o$ vanish at p

$$\frac{\partial^{i_1+\dots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(p) = 0 \qquad \text{with } i_1 + \dots + i_n < o$$

and if there exists at least one non-vanishing o-th partial derivative

$$\frac{\partial^{i_1+\dots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(p) \neq 0 \qquad \text{with } i_1 + \dots + i_n = o.$$

Rationalising square roots



Points of multiplicity 1 are called **regular points**,
points of multiplicity $o > 1$ are called **singular points** of V.

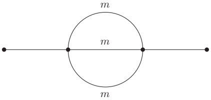
Theorem

Let $f(x_1, \ldots, x_n)$ be a polynomial of degree d. If $V(f)$ has a point of multiplicity $(d-1)$, the square root $\sqrt{f(x_1, \ldots, x_n)}$ can be rationalised.

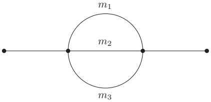
Section 5

Elliptic Feynman integrals

Motivation



- Two-loop equal mass sunrise
  - 3 master integrals
  - 1 kinematic variable



- Two-loop unequal mass sunrise
  - 7 master integrals
  - 3 kinematic variable

The equal mass sunrise

With $\vec{I} = (I_{110}, I_{111}, I_{211})^T$, $x = -p^2/m^2$ and $\mu^2 = m^2$ we have the differential equation $(d + A)\vec{I} = 0$ with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(D-3) & -3 \\ 0 & \frac{1}{6}(D-3)(3D-8) & \frac{1}{2}(3D-8) \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{8}(D-3)(3D-8) & -(D-3) \end{pmatrix} \frac{dx}{x+1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{24}(D-3)(3D-8) & -(D-3) \end{pmatrix} \frac{dx}{x+9}.$$

Subsection 1

Background from Mathematics

Algebraic curves

- Ground field $\mathbb{C}$
- **Algebraic curve** in $\mathbb{C}^2$ **defined by** a **polynomial** $P(x,y)$:

$$P(x,y) = 0$$

- Projective space $\mathbb{CP}^2$ with homogeneous coordinates $[x:y:z]$:
  Algebraic curve in $\mathbb{CP}^2$ defined by a homogeneous polynomial $P(x,y,z)$:

$$P(x,y,z) = 0$$

  We usually work in the chart $z=1$.

Elliptic curves

Definition (Elliptic curve over $\mathbb{C}$)

An algebraic curve in $\mathbb{CP}^2$ of genus one with one marked point.

Example (Weierstrass normal form)

In the chart $z = 1$:

$$y^2 = 4x^3 - g_2 x - g_3$$

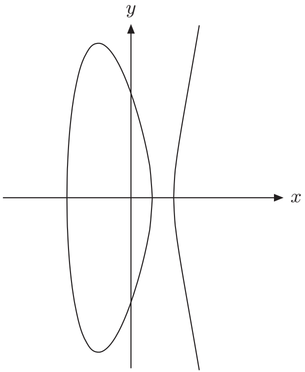
Example (Quartic form)

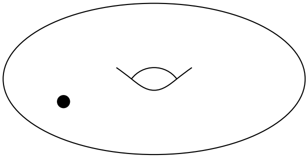
In the chart $z = 1$:

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

Riemann surfaces

One complex dimension corresponds to two real dimensions.





Weierstrass normal form
$y^2 = 4x^3 - g_2 x - g_3$

Real Riemann surface of genus one with one marked point

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A **period** ψ of the function f is a constant such that for all z:

$$f(z+\psi) = f(z)$$

The set of all periods of f forms a **lattice**, which is either

- trivial (i.e. the lattice consists of $\psi = 0$ only),
- a simple lattice, $\Lambda = \{n\psi \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\psi_1 + n_2\psi_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Double periodic functions are called **elliptic functions**.

Examples of periodic functions

- Singly periodic function: **Exponential function**

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\psi = 2\pi i$.

- Doubly periodic function: **Weierstrass's $\wp$-function**

$$\wp(z) = \frac{1}{z^2} + \sum_{\psi \in \Lambda \backslash \{0\}} \left(\frac{1}{(z+\psi)^2} - \frac{1}{\psi^2} \right), \qquad \Lambda = \{ n_1 \psi_1 + n_2 \psi_2 | n_1, n_2 \in \mathbb{Z} \},$$

$$\mathrm{Im}(\psi_2/\psi_1) \neq 0.$$

$\wp(z)$ is periodic with periods ψ_1 and ψ_2.

Inverse functions

The corresponding inverse functions are in general **multivalued functions**.

- For the exponential function $x = \exp(z)$ the inverse function is the **logarithm**

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an **elliptic integral**

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\psi \in \Lambda \backslash \{0\}} \frac{1}{\psi^4}, \quad g_3 = 140 \sum_{\psi \in \Lambda \backslash \{0\}} \frac{1}{\psi^6}.$$

Elliptic integrals

Complete elliptic integrals

- First kind:

$$K(x) = \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

- Second kind:

$$E(x) = \int\limits_0^1 dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}$$

- Third kind:

$$\Pi(v,x) = \int\limits_0^1 \frac{dt}{(1-vt^2)\sqrt{(1-t^2)(1-x^2t^2)}}$$

Incomplete elliptic integrals

- First kind:

$$F(z,x) = \int\limits_0^z \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

- Second kind:

$$E(z,x) = \int\limits_0^z dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}$$

- Third kind:

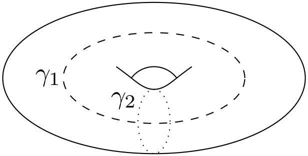
$$\Pi(v,z,x) = \int\limits_0^z \frac{dt}{(1-vt^2)\sqrt{(1-t^2)(1-x^2t^2)}}$$

Abelian differentials

- Abelian differential of the first kind:
  holomorphic
- Abelian differential of the second kind:
  meromorphic with **all residues vanishing**
- Abelian differential of the third kind:
  meromorphic with **non-zero residues**

Periods of an elliptic curve

Integrate the **holomorphic differential** along the two independent cycles.



Periods of an elliptic curve

Example

The Legendre form:

$$y^2 = x(x-1)(x-\lambda)$$

The periods are

$$\psi_1 = 2 \int\limits_0^{\lambda} \frac{dx}{y} = 4K\left(\sqrt{\lambda}\right) \qquad \psi_2 = 2 \int\limits_1^{\lambda} \frac{dx}{y} = 4iK\left(\sqrt{1-\lambda}\right)$$

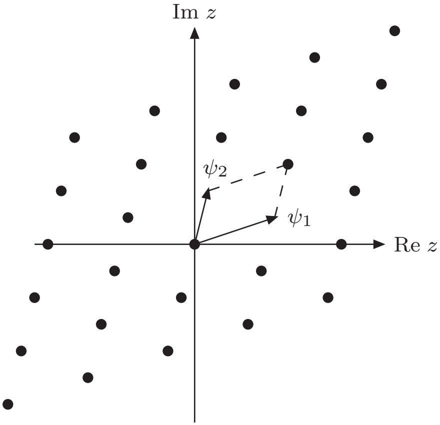
The elliptic curve $y^2 = x(x-1)(x-\lambda)$ depends on a parameter λ, and so do the periods $\psi_1(\lambda)$ and $\psi_2(\lambda)$.

How do the periods change, if we change λ?

The variation is governed by a second-order differential equation:
With $t = \sqrt{\lambda}$ we have

$$\underbrace{\left[t\left(1-t^2\right)\frac{d^2}{dt^2} + \left(1-3t^2\right)\frac{d}{dt} - t \right]}_{\text{Picard-Fuchs operator}} \psi_j \quad = \quad 0$$

Representing an elliptic curve as $\mathbb{C}/\Lambda$



Points inside fundamental parallelogram $\Leftrightarrow$ Points on elliptic curve

Back and forth

- **Weierstrass normal form** $\rightarrow \mathbb{C}/\Lambda$:

  Given a point (x,y) with $y^2 - 4x^3 + g_2 x + g_3 = 0$ the corresponding point $z \in \mathbb{C}/\Lambda$ is given by

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}$$

- $\mathbb{C}/\Lambda \rightarrow$ **Weierstrass normal form**:

  Given a point $z \in \mathbb{C}/\Lambda$ the corresponding point (x,y) on $y^2 - 4x^3 + g_2 x + g_3 = 0$ is given by

$$(x,y) = \left(\wp(z), \wp'(z)\right)$$

Notation

Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$, where

$$\tau = \frac{\psi_2}{\psi_1}$$

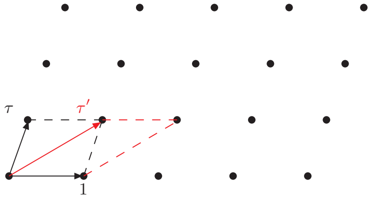
and require $\mathrm{Im}(\tau) > 0$.

Definition (The complex upper half-plane)

$$\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$$

Modular transformations

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

1. f transforms under modular transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

2. f is holomorphic on $\mathbb{H}$,
3. f is holomorphic at $i\infty$.

Define the $|_k\gamma$ operator by

$$(f|_k\gamma)(\tau) \quad = \quad (c\tau+d)^{-k} \cdot f(\gamma(\tau))$$

Congruence subgroups

Apart from $\mathrm{SL}_2(\mathbb{Z})$ we may also look at congruence **subgroups**, for example

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; b, c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "**nice**" transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Modular forms

For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ denote by $\mathcal{M}_k(\Gamma)$ the **space of modular forms of weight k**.
We have the inclusions

$$\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathcal{M}_k(\Gamma_0(N)) \subseteq \mathcal{M}_k(\Gamma_1(N)) \subseteq \mathcal{M}_k(\Gamma(N))$$

For $f \in \mathcal{M}_k(\Gamma(N))$:

$$\begin{aligned} f|_k\gamma &= f, & \gamma &\in \Gamma(N) \\ f|_k\gamma &\in \mathcal{M}_k(\Gamma(N)), & \gamma &\in \mathrm{SL}_2(\mathbb{Z})\backslash\Gamma(N) \end{aligned}$$

Notation

For $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ set

$$\bar{q} = \exp(2\pi i \tau), \qquad \bar{w} = \exp(2\pi i z)$$

Maps the complex **upper half-plane** $\tau \in \mathbb{H}$ **to** the **unit disk** $|\bar{q}| < 1$.

Trivialises periodicity with period 1:

$$\bar{q}(\tau+1) = \bar{q}(\tau), \qquad \bar{w}(z+1) = \bar{w}(z)$$

Shifts with τ correspond to multiplication with $\bar{q}$:

$$\bar{q}(\tau+\tau) = \bar{q}(\tau) \cdot \bar{q}(\tau), \qquad \bar{w}(z+\tau) = \bar{w}(z) \cdot \bar{q}(\tau)$$

Iterated integrals of modular forms

Let $f_1, \dots, f_n$ be modular forms.

$$I(f_1, f_2, ..., f_n; q) = (2\pi i)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1(\tau_1) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2(\tau_2) \dots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n(\tau_n)$$

As basepoint we usually take $\tau_0 = i\infty$.

An integral over a modular form is in general **not** a modular form.

Analogy: An integral over a rational function is in general not a rational function.

Simple poles at $\tau = i\infty$

A modular form $f_k(\tau)$ is by definition holomorphic at the cusp and has a $\bar{q}$-expansion

$$f_k(\tau) = a_0 + a_1 \bar{q} + a_2 \bar{q}^2 + ..., \qquad \bar{q} = \exp(2\pi i \tau)$$

The transformation $\bar{q} = \exp(2\pi i \tau)$ transforms the point $\tau = i\infty$ to $\bar{q} = 0$ and we have

$$2\pi i \, f_k(\tau) d\tau = \frac{d\bar{q}}{\bar{q}} \left(a_0 + a_1 \bar{q} + a_2 \bar{q}^2 + ... \right).$$

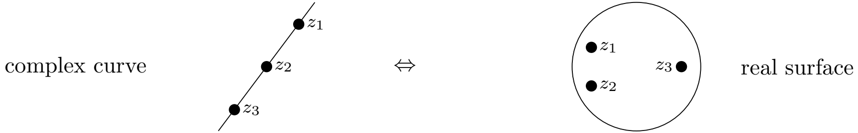
Thus a modular form non-vanishing at the cusp $\tau = i\infty$ has a **simple pole** at $\bar{q} = 0$.

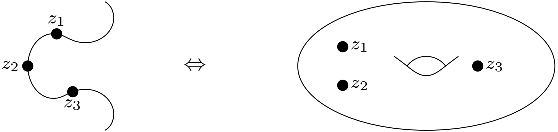
Subsection 2

Moduli spaces

Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.





Coordinates

Genus 0: dim $\mathcal{M}_{0,n} = n - 3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, \dots, \mathbf{z_{n-3}})$

Genus 1: dim $\mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, \dots, \mathbf{z_{n-1}})$

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j.$$

The iterated integral is defined by

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) \quad = \quad \int\limits_0^\lambda d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Iterated integrals on $\mathcal{M}_{0,n}$

We are interested in differential one-forms, which have only simple poles:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{dy}{y - z_j}.$$

Multiple polylogarithms:

$$G(z_1, \dots, z_k; y) = \int_0^y \frac{dy_1}{y_1 - z_1} \int_0^{y_1} \frac{dy_2}{y_2 - z_2} \dots \int_0^{y_{k-1}} \frac{dy_k}{y_k - z_k}, \qquad z_k \neq 0$$

Iterated integrals on $\mathcal{M}_{1,n}$

- Coordinates are $(\tau, z_1, ..., z_{n-1})$
- Decompose an arbitrary path along $d\tau$ and dz_j
- Two classes of iterated integrals:
  1. Integration along z
  2. Integration along τ
- What are the differential one-forms we want to integrate?

The Kronecker function

The **first Jacobi theta function** $\theta_1(z,q)$:

$$\theta_1(z,q) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{i(2n+1)z}, \qquad q = e^{i\pi\tau}$$

The **Kronecker function** $F(z,\alpha,\tau)$:

$$F(z,\alpha,\tau) = \pi\theta_1'(0,q) \frac{\theta_1(\pi(z+\alpha),q)}{\theta_1(\pi z,q)\,\theta_1(\pi\alpha,q)} = \frac{1}{\alpha} \sum_{k=0}^{\infty} g^{(k)}(z,\tau) \alpha^k$$

We are mainly interested in the coefficients $g^{(k)}(z,\tau)$ of the Kronecker function.

The coefficients $g^{(k)}(z,\tau)$ of the Kronecker function

Properties of $g^{(k)}(z,\tau)$:

1. **only simple poles** as a function of z
2. **quasi-periodic** as a function of z: Periodic by 1, quasi-periodic by τ.

$$\begin{aligned} g^{(k)}(z+1,\tau) &= g^{(k)}(z,\tau), \\ g^{(k)}(z+\tau,\tau) &= \sum_{j=0}^{k} \frac{(-2\pi i)^j}{j!} g^{(k-j)}(z,\tau) \end{aligned}$$

3. **almost modular**:

$$g^{(k)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \sum_{j=0}^{k} \frac{(2\pi i)^j}{j!} \left(\frac{cz}{c\tau+d}\right)^j g^{(k-j)}(z,\tau)$$

Differential one-forms on $\mathcal{M}_{1,n}$

- To keep the discussion simple, we start with $\mathcal{M}_{1,2}$ with coordinates (τ, z):
  - One-forms from modular forms:

$$\omega_k^{\mathrm{modular}} = 2\pi i \, f_k(\tau) d\tau$$

  - One-forms from the Kronecker function:

$$\omega_k^{\mathrm{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(z - c_j, \tau) \, dz + (k-1) \, g^{(k)}(z - c_j, \tau) \frac{d\tau}{2\pi i} \right]$$

    with c_j being a constant.
- We allow the substitution $\tau \to K\tau$ with $K \in \mathbb{N}$.
- On $\mathcal{M}_{1,n}$ with coordinates $(\tau, z_1, ..., z_{n-1})$ we consider $z \to z_j$ with $1 \leq j \leq (n-1)$.

Iterated integrals on $\mathcal{M}_{1,n}$: Integration along z

Differential one-forms:

$$\omega_k^{\mathrm{Kronecker},z}(z_j,\tau) = (2\pi i)^{2-k}\, g^{(k-1)}(z-z_j,\tau)\, dz$$

Elliptic multiple polylogarithms:

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_r \\ z_1 & \dots & z_r \end{smallmatrix}; z; \tau\right) = (2\pi i)^{n_1+\dots+n_r-r}\, I\left(\omega_{n_1+1}^{\mathrm{Kronecker},z}(z_1,\tau), \dots, \omega_{n_r+1}^{\mathrm{Kronecker},z}(z_r,\tau); z\right)$$

- $\tau = \mathrm{const}$
- meromorphic version, only simple poles in z
- not double periodic!

Iterated integrals on $\mathcal{M}_{1,n}$: Integration along τ

Differential one-forms:

$$
\begin{aligned}
\omega_k^{\mathrm{Kronecker},\tau}(z_j) &= (2\pi i)^{2-k}\,(k-1)\,g^{(k)}(z_j,\tau)\,\frac{d\tau}{2\pi i} \\
&= \frac{(k-1)}{(2\pi i)^k}\,g^{(k)}(z_j,\tau)\,\frac{d\bar{q}}{\bar{q}}
\end{aligned}
$$

- Integrate in $\bar{q}$
- No poles in $0 < |\bar{q}| < 1$.
- Possibly a simple pole at $\bar{q} = 0$ (“trailing zero”)

Subsection 3

Physics

The equal-mass sunrise

It is **not possible** to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.
However by factoring off the (**non-algebraic**) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 I_{110}\left(2-2\varepsilon, x\right) \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} I_{111}\left(2-2\varepsilon, x\right) \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2$$

If in addition one makes a (**non-algebraic**) change of variables from x to τ, one obtains

$$\frac{d}{d\tau} I = \varepsilon \, A(\tau) \, I,$$

where $A(\tau)$ is an ε-independent 3×3-matrix whose **entries are modular forms**.

The unequal-mass sunrise

After a redefinition of the basis of master integrals and a change of coordiantes from $(x, y_1, y_2) = (p^2/m_3^2, m_1^2/m_3^2, m_2^2/m_3^2)$ to (τ, z_1, z_2) one finds

$$\mathbf{A} = \varepsilon \sum_{\mathbf{j}=1}^{\mathbf{N_L}} \mathbf{C_j} \, \omega_{\mathbf{j}},$$

where ω_j is either

$$2\pi i \, f_k(\tau) \, d\tau,$$

where $f_k(\tau)$ is a modular form, or of the form

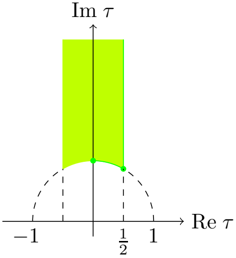
$$\omega_k(z_i, K\tau) = (2\pi i)^{2-k} \left[g^{(k-1)}(z_i, K\tau) \, dz_i + K(k-1) \, g^{(k)}(z_i, K\tau) \frac{d\tau}{2\pi i} \right]$$

$\bar{q}$-expansions

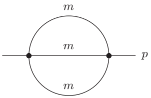
- Iterated integrals in the elliptic case are evaluated with the help of their $\bar{q}$-expansions, $\bar{q} = \exp(2\pi i \tau)$.
- The $\bar{q}$-series converge for $|\bar{q}| < 1$.
- By a modular transformation we may map τ to the fundamental domain, resulting in

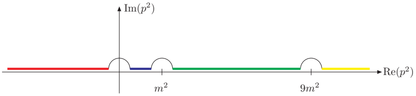
$$|\bar{q}| \quad \leq \quad e^{-\pi\sqrt{3}} \approx 0.0043,$$

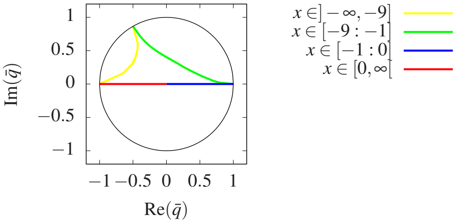
resulting in a fast converging series.



- Consider the equal mass sunrise integral with $x = -p^2/m^2$.
- Singularites at $x \in \{-9, -1, 0, \infty\}$.
- In the variable x we don't expect an expansion around one singular point to converge beyond the next singular point.
- In the variable $\bar{q}$ the expansion converges for all values $x \in \mathbb{R}$ except the three other singular points.







Numerics

Physics is about numbers:

- Iterated integrals of modular forms and elliptic multiple polylogarithms can be evaluated numerically with arbitrary precision.
- Implemented in `GiNaC`.

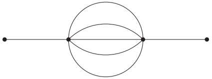
Walden, S.W, '20

```
ginsh - GiNaC Interactive Shell (GiNaC V1.8.1)
  __,  _______  Copyright (C) 1999-2021 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type `warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> iterated_integral({Eisenstein_kernel(3,6,-3,1,1,2)},0.1);
0.23675657575197179243274817775862177623438999192840338805367
```

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of higher genus g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and higher dimensional objects: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.