Deep Inelastic Scattering

16 January, 2024

Kay Schönwald

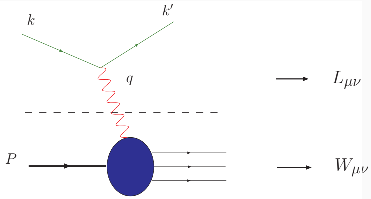
Bhubaneswar, January 16, 2024

University of Zürich

Outline

Introduction

Theory of Deep Inelastic Scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2\mathrm{d}x} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi} \int \mathrm{d}^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{\mathrm{em}}(\xi), J_\nu^{\mathrm{em}}(\xi) \right] | P \rangle \\ &= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \\ &+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot P S^\sigma - q \cdot S P^\sigma \right)}{(q \cdot P)^2} g_2(x, Q^2) \end{aligned}$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z)\ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x)\ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j\,|O_i|\,j\rangle|\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Massless Wilson Coefficients

Massless Wilson Coefficients – Operator Product Expansion

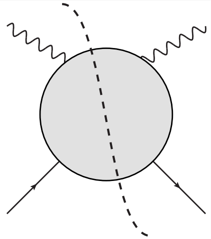
$$T_{\mu\nu} = \sum_{N,j} \left(\frac{1}{x}\right)^N \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) C_{L,j}^N + \left(-g_{\mu\nu} - \frac{4x^2}{q^2} p_\mu p_\nu - \frac{2x}{q^2} (p_\mu q_\nu + p_\nu q_\mu) \right) C_{2,j}^N \right] A_{P,N}^j$$

- We find an expansion for unphysical x ($x \to \infty$), which defines Mellin moments.
- The hadronic matrix elements $A_{P,N}^j$ are related to (moments) of the parton densities.
- For the calculation of the perturbative Wilson coefficients we use partonic states.
  $\Rightarrow$ Then all loop corrections to the matrix elements vanish.

$$\langle i \middle| O^{j,\{\mu_1,\ldots,\mu_N\}} \middle| j \rangle \sim \delta_{i,j}, \qquad i,j = \mathrm{q,g}$$

Massless Wilson Coefficients

- The massless Wilson coefficients can be calculated by evaluating the forward compton amplitude.
- Moments of the Wilson coefficients can be calculated by the (unphysical) expansion $x \to \infty$.



Status (unpolarized):

- NLO: [Furmanski, Petronzio '82; ...]
- NNLO: [van Neerven, Zijlstra '91,'92; ... ; Moch, Vermaseren '00]
- N^{3}LO: [Moch, Vermasern, Vogt, Nucl.Phys.B '05,'09'; Moch, Rogal, Vogt '08; Blümlein, Marquard, Schneider Schönwald '22]
- N^{4}LO (n_f^2 in the non singlet case): [Basdew-Sharma, Pelloni, Herzog, Vogt '22]

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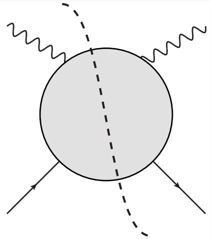

Status (moments):

- N^3LO: [Larin, van Ritbergen, Vermaseren '94; Larin '97; Retey, Vermaseren '00; Blümlein, Vermaseren '05; Moch, Rogal '07]
- N^4LO (non singlet case): [Ruijl, Ueda, Vermaseren, Davies, Vogt '16; Moch, Ruijl, Ueda, Vermaseren, Vogt '22;]

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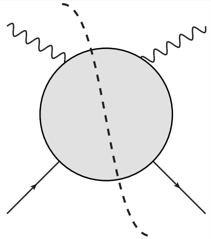


Status (polarized):

- NLO: [Furmanski, Petronzio '82; ...]
- NNLO: [van Neerven, Zijlstra '94; Vogt, Moch, Rogal, Vermaseren '08; ...]
- N^3LO: [Blümlein, Marquard, Schneider, Schönwald '22]

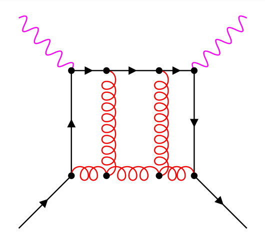
Moments of the Massless Wilson Coefficients

- The calculation of fixed moments is simpler.
- The expansion $x \to \infty$ can be implemented via a naive Taylor expansion in the momentum p:

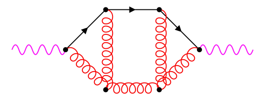
$$\frac{1}{(k_i + p)^2} \to \sum_{i=0}^{\infty} \frac{(2k_i \cdot p)^i}{(k_i^2)^{i+1}}$$

- In this way the Feynman diagrams reduce to massless 2-point functions.
- There are very powerful tools for the reduction of this class of Feynman diagrams:
  - up to 3-loop mincer [Larin, Tkachov, Vermaseren '91]
  - up to 4-loop forcer [Ruijl, Ueda, Vermaseren '17]
- Calculation of high moments rely on the efficient reduction techniques.

Moments of the Massless Wilson Coefficients



$\overset{p\to 0}{\Longrightarrow}$

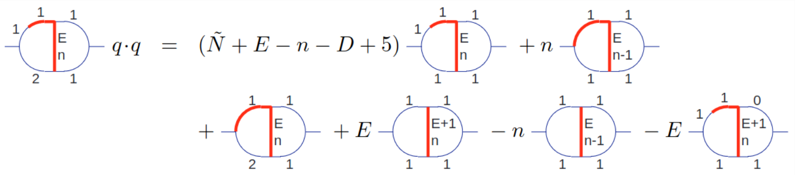


$$\times \prod_{j=1}^{3} \frac{1}{l_j^2} \sum_{i_j=1}^{\infty} \left(\frac{2p \cdot l_j}{l_j^2} \right)^{i_j}$$

# of loops	# of topologies	# of masters
1	1	1
2	1	2
3	3	6
4	11	28

All-N results for the Massless Wilson Coefficients

- **First NLO and NNLO calculations:**
  - Calculate phase space integrals in momentum space.
  - Covert to Mellin space afterwards.
- **NNLO and N^{3}LO calculations** [Moch, Vermaseren, Vogt '05]
  - Hunt for recursion relations using integration-by-parts, scaling relations etc. .
  - Solve the recursion relations to obtain the N-space result.



All-N results for the Massless Wilson Coefficients

Our calculation:

- Mellin moments can be calculated by expansion in $y = \frac{1}{x} = \frac{2p.q}{Q^2}$: $M[F_i](N) = \frac{1}{N!}[\frac{d^N F_i}{dy^N}]_{y=0}$.
- Diagram generation: `QGRAF` [Nogueira '91]
- Lorentz, Dirac and color algebra: `TFORM` [Ruijl, Ueda, Vermaseren, Tentyukov '17] with `color.h` [Ritbergen, Schellekens, Vermaseren '99]
- Match to a minimal number of topologies.
- IBP reduction and differential equations: `Crusher` [Marquard, Seidel]

# of loops	# diagrams	# of topologies	# of integrals	# of masters
0	2	0	0	0
1	8	2	26	3
2	126	6	490	20
3	2906	61	70248	293
4	85199	700	$\sim 10^7$	3000?

Massless Wilson Coefficients

Technical Aspects:
We use two independent methods to compute the Mellin space result:

1. Method of large moments:
    - Compute a large number of moments: `SolveCoupledSystems` [Blümlein, Schneider '17]
    - Determine a recurrence from the moments: `Guess` [Kauers '09-'15]
    - Solve the recurrence: `Sigma` [Schneider '07]
2. Analytic computation of the master integrals: [Ablinger, Blümlein, Marquard, Rana, Schneider '18]
    - Decouple systems of differential equations: `OreSys` [Gerhold, Schneider '02]
    - Solve analytically in y via factorization of the differential operator: `HarmonicSums` [Ablinger '10 -]
    - Take the Nth derivative symbolically to obtain the Mellin space expression: `HarmonicSums`

Massless Wilson Coefficients

Method 1: Method of large moments

- We calculated a large number of moments: $F_i = \sum_{j=0}^{\infty} C_i^{(j)} y^i$

$$-2, 0, -\frac{1}{6}, -\frac{1}{6}, -\frac{3}{20}, -\frac{2}{15}, -\frac{5}{42}, -\frac{3}{28}, -\frac{7}{72}, -\frac{4}{45}, -\frac{9}{110}, -\frac{5}{66}, -\frac{11}{156}, -\frac{6}{91}, \\ -\frac{13}{210}, -\frac{7}{120}, -\frac{15}{272}, -\frac{8}{153}, -\frac{17}{342}, -\frac{9}{190}, -\frac{19}{420}, \dots$$

- We can guess a recurrence:

$$N^2 C_N - (N-1)(N+2) C_{N+1} = 0$$

- We can solve the recurrence:

$$C_N = -\frac{N-1}{N(N+1)}, \quad N > 0$$

→ We directly obtain the result in Mellin-space.

Massless Wilson Coefficients

- We calculated a large number of moments!

Wilson coefficient	1 loop	2 loop	3 loop
F_1^{NS}	126	1219	4300
F_1^{PS}	0	374	1708
F_1^{g}	104	960	3534
F_L^{NS}	48	560	2387
F_L^{PS}	0	175	774
F_L^{g}	54	434	2046
xF_3^{NS}	126	1219	4171
g_1^{NS}	126	1219	4171
g_1^{PS}	0	175	1458
g_1^{g}	84	1166	2998

Massless Wilson Coefficients

Method 2: Analytic computation of the master integrals

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -\frac{2(1+\epsilon)}{y} & \frac{2(1+\epsilon)}{(1+4\epsilon)y} \\ -\frac{(1+4\epsilon)^2}{y(1-y)} & \frac{1+4\epsilon+(1-\epsilon)y}{y(1-y)} \end{pmatrix} \cdot \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} + \vec{R}(y,\epsilon),$$

where $\vec{R}(y,\epsilon)$ are contributions from easier integrals.

- We want to study the solutions of the system (in $D = 4 - 2\epsilon = 4$).
- We can uncouple the differential equation to a second order one for either M_1 or M_2 and obtain:

$$M_1''(y) + \frac{2(1-2y)}{y(1-y)} M_1'(y) - \frac{2}{y(1-y)} M_1(y) = r(y)$$

- This differential equation factorizes:

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} + \frac{2-3y}{y(1-y)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}y} - \frac{1}{1-y}\right) M_1(y) = r(y)$$

Massless Wilson Coefficients

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and define

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} - \frac{1}{1-y}\right) h_1(y) = 0, \qquad \left(\frac{\mathrm{d}}{\mathrm{d}y} + \frac{2-3y}{y(1-y)}\right) h_2(y) = 0,$$

we can obtain the solutions via

$$f_1(y) = h_1(y) = -\frac{1}{1-y}$$

$$f_2(y) = h_1(y) \int_0^y dy_1 \frac{h_2(y_1)}{h_1(y_1)} = \frac{1}{y(1-y)}$$

$$f_{part}(y) = h_1(y) \int_0^y dy_1 \frac{h_2(y_1)}{h_1(y_1)} \int_0^{y_1} dy_2 \frac{r(y_2)}{h_2(y_2)}$$

This can be generalized to differential operators which factorize arbitrary many linear factors.
→ We obtain the generating function, with $M[f](N) = \frac{1}{N!}[\frac{d^N F_i}{dy^N}]_{y=0}$.

Mellin-Space – Relations between different spaces

$$\hat{f}(t) = \sum_{N=1}^{\infty} t^N \tilde{f}(N)$$

$$\updownarrow$$

$$\tilde{f}(N) = \int_0^1 \mathrm{d}x\, x^{N-1}\, f(x)$$

$$\updownarrow$$

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-N} \tilde{f}(N) \mathrm{d}N$$

- $\hat{f}(t) \to \tilde{f}(N)$ and $\hat{f}(x) \to \tilde{f}(N)$: calculable via recurrence equations
- $\tilde{f}(N) \to f(x)$: calculable via differential equations
- $\hat{f}(t) \to f(x)$: calculable via analytic continuation

but: algorithmic solution only possible if recurrences or differential equations factorize to first order

Mellin-Space – Relations between different spaces

$$\hat{f}(t) = \sum_{N=1}^{\infty} \tilde{f}(N) t^N = \sum_{N=1}^{\infty} \int_0^1 \mathrm{d}x' \; t^N {x'}^{N-1} f(x') = \int_0^1 \mathrm{d}x' \; \frac{t}{1 - tx'} f(x')$$

Setting $t = \frac{1}{x}$ we obtain:

$$\hat{f}\left(\frac{1}{x}\right) = \int_0^1 \mathrm{d}x' \frac{f(x')}{x - x'}$$

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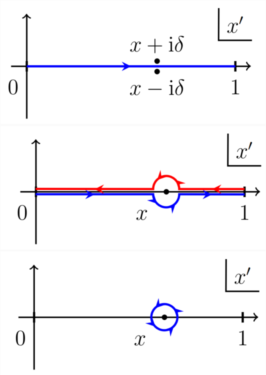

Setting $t = \frac{1}{x}$ we obtain:

$$\hat{f}\left(\frac{1}{x}\right) = \int_0^1 \mathrm{d}x' \frac{f(x')}{x-x'}$$

Therefore:

$$f(x) = \frac{i}{2\pi} \lim_{\delta \to 0} \oint_{|x-x'|=\delta} \frac{f(x')}{x-x'} = \frac{i}{2\pi} \underset{x}{\mathrm{Disc}} \; \hat{f}\left(\frac{1}{x}\right)$$

Inverse Mellin transform via analytic continuation

The discussion before used some implicit assumptions.
The x-space representation

1. has no $(-1)^N$ term.
2. is regular and has now contributions from distributions.
3. has a support only on $x \in (0, 1)$.

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For physical examples:

$$\tilde{f}(N) = \int_0^1 \mathrm{d}x\, x^{N-1} \left[f(x) + (-1)^N g(x) + \left(f_\delta + (-1)^N g_\delta \right) \delta(1-x) \right] + \int_0^1 \mathrm{d}x\, \frac{x^{N-1} - 1}{1-x} \left[f_+(x) + (-1)^N g_+(x) \right]$$

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All of this can be lifted, but the discussion is more involved.

The Wilson coefficients can be expressed by:

- harmonic sums in Mellin space

$$S_{a,\vec{b}}(N) = \sum_{i=1}^{N} \frac{\text{sign}(a)^i}{i^a} S_{\vec{b}}(i)$$

- harmonic polylogarithms in momentum space

$$H_{a,\vec{b}}(x) = \int_0^x \mathrm{d}x' f_a(x') H_{\vec{b}}(x')$$

with $f_0(x) = \frac{1}{x}$, $f_1(x) = \frac{1}{1-x}$, $f_{-1}(x) = \frac{1}{1+x}$

[Vermaseren '99; Blümlein, Kurth '99; Remiddi, Vermaseren '00]

Massive Wilson Coefficients

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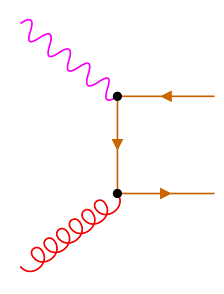


We have to calculate the process:

$$q(p)+\gamma^*(q)\to Q(k_1)+Q(k_2),$$

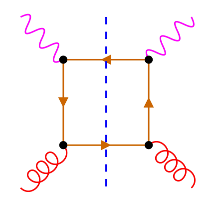
with $q^2=-Q^2,\ p^2=0,\ k_1^2=k_2^2=m^2,\ (p+q)^2=s=Q^2(1-z)/z,\ \beta=\sqrt{1-4m^2/s}$

- Parametrize the phase space:
  $p=\frac{s-Q^2}{2\sqrt{s}}(1,0,0,1),\ k_1=\frac{\sqrt{s}}{2}(1,0,\beta\cos\theta,\beta\sin(\theta)),\ k_2=\frac{\sqrt{s}}{2}(1,0,-\beta\cos\theta,-\beta\sin(\theta))$

$$\int \mathrm{d}\mathrm{PS}_2=\int\frac{\mathrm{d}^4k_1}{(2\pi)^d}\int\frac{\mathrm{d}^4k_2}{(2\pi)^d}(2\pi)^d\delta^{(d)}(p+q-k_1-k_2)(2\pi)^2\delta(k_1^2-m^2)\delta(k_2^2-m^2)$$

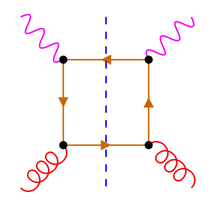
$$=2^{4-2d}\frac{\pi^{1-d/2}}{\Gamma(d/2-1)}s^{d/2-2}\beta^{d-3}\int_0^\pi \mathrm{d}\theta\sin^{d-3}(\theta)$$

Massive Wilson Coefficients



$$\sim \int \mathrm{dPS}_2 \frac{T_F \, \mathrm{tr}\left[(\not{k}_1 + m)\gamma_\mu(\not{k}_1 - \not{p} + m)\gamma_\nu(-\not{k}_2 + m)\gamma_\rho(\not{k}_1 - \not{p} + m)\gamma_\sigma\right]}{\left[(k_1 - p)^2 - m^2\right]^2}$$

Massive Wilson Coefficients



$$\sim \int \mathrm{dPS}_2 \frac{T_F \,\mathrm{tr}\left[(\not{k}_1 + m)\gamma_\mu(\not{k}_1 - \not{p} + m)\gamma_\nu(-\not{k}_2 + m)\gamma_\rho(\not{k}_1 - \not{p} + m)\gamma_\sigma\right]}{\left[(k_1 - p)^2 - m^2\right]^2}$$

$$\sim \int_0^\pi \mathrm{d}\theta \frac{\sin^{d-j}(\theta)}{1 - \beta \sin(\theta)}$$

Massive Wilson Coefficients

Leading Order: $F_{2,L}(x, Q^2)$

[Witten '76, Babcock, Siver '78, Shifman, Vainshtein, Zakharov '78, Leveille, Weiler '79, Glück, Reya '79, Glück, Hoffmann, Reya '82]

$$F_{2,L}^{\mathrm{LO}}(x, Q^2) = e_Q^2 a_s(Q^2) \int_{ax}^{1} \frac{\mathrm{d}y}{y} C_{F_2(F_L)}^{(1)}(x/y, m_Q^2, Q^2) f_g(y, Q^2)$$

$$C_{F_2}^{(1)}(z, m_Q^2, Q^2) = 8 T_F \Bigg\{ \beta \left[-\frac{1}{2} + 4z(1-z) + 2\frac{m_Q^2}{Q^2} z(z-1) \right] + \left[-\frac{1}{2} + z(1-z) + 2\frac{m_Q^2}{Q^2} z(3z-1) + 4\frac{m_Q^4}{Q^4} z^2 \right] \ln\left(\frac{1-\beta}{1+\beta} \right) \Bigg\}$$

$$C_{F_L}^{(1)}(z, m_Q^2, Q^2) = 16 T_F \left[z(1-z)\beta + 2\frac{m_Q^2}{Q^2} z^2 \right] \ln\left(\frac{1-\beta}{1+\beta} \right)$$

with $a = 1 + 4m_Q^2/Q^2$, $\beta = \sqrt{1 - \frac{4m_Q^2 z}{Q^2(1-z)}}$

Massive Wilson Coefficients

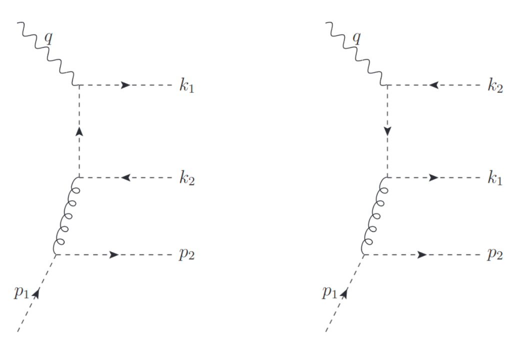
Next-To-Leading Order: $F_{2,L}(x, Q^2)$

[Laenen, Riemersma, Smith, van Neerven '93,'95, Blümlein, Falcioni, De Freitas '16,Blümlein, Raab, Schönwald '19]

$$F_{2,L}^{\mathrm{NLO}}(x,Q^2) = \frac{Q^2}{\pi m_Q^2}\alpha_s^2 \int_{xa}^{1} \frac{\mathrm{d}y}{y} \Bigg\{ e_Q^2 f_g\left(\frac{x}{y}, Q^2\right)\left(c_{k,g}^{(1)}(\xi,\eta) + \bar{c}_{k,g}^{(1)}(\xi,\eta)\ln\left(\frac{Q^2}{m^2}\right)\right)$$
$$+ \sum_{i=q,\bar{q}} \Bigg[e_Q^2 f_i\left(\frac{x}{y}, Q^2\right)\left(c_{k,i}^{(1)}(\xi,\eta) + \bar{c}_{k,i}^{(1)}(\xi,\eta)\ln\left(\frac{Q^2}{m^2}\right)\right)$$
$$+ e_i^2 f_i\left(\frac{x}{y}, Q^2\right)\left(d_{k,i}^{(1)}(\xi,\eta) + \bar{d}_{k,i}^{(1)}(\xi,\eta)\ln\left(\frac{Q^2}{m^2}\right)\right)\Bigg]\Bigg\}$$

- Semi-analytic expressions known (not all integrals could be done analytically).
- Semi-analytic Mellin space expressions available.

Massive Wilson Coefficients – Pure-Singlet



$$t = 2p_1 \cdot p_2, \qquad u = 2p_2 \cdot q,$$
$$s = (p_1 + q)^2, \qquad s_{12} = (k_1 + k_2)^2.$$

$$\begin{aligned}
k_1 &= \left(k^0, 0, \ldots, |\vec{k}|\sin(\phi)\sin(\theta), |\vec{k}|\cos(\phi)\sin(\theta), |\vec{k}|\cos(\theta)\right),\\
k_2 &= \left(k^0, 0, \ldots, -|\vec{k}|\sin(\phi)\sin(\theta), -|\vec{k}|\cos(\phi)\sin(\theta), -|\vec{k}|\cos(\theta)\right),\\
p_1 &= \frac{s - t - q^2}{2\sqrt{s_{12}}}(1, \ldots, 0, 0, 1),\\
p_2 &= \frac{s - s_{12}}{2\sqrt{s_{12}}}(1, 0, \ldots, \sin(\chi), \cos(\chi)),\\
q &= \frac{1}{2\sqrt{s_{12}}}\left(q^2 + s_{12} + t, \ldots, 0, 0, (s - s_{12})\sin(\chi), q^2 + t - s + (s - s_{12})\cos(\chi)\right)
\end{aligned}$$

$$\int \mathrm{dPS}_3 = \left[\prod_{i=1}^{3} \int \frac{\mathrm{d}^4 k_i}{(2\pi)^d}(2\pi)\delta(k_i^2 - m^2)\right](2\pi)^d \delta^{(d)}(p_1 + q - k_1 - k_2 - k_3)$$

$$\sim \int_{s_{12}^-}^{s_{12}^+} \mathrm{d}s_{12} \int_{t^-}^{t^+} \mathrm{d}t \int_0^{\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \, [\sin(\theta)]^{d-3} [\sin(\phi)]^{d-4} s_{12}^{d/2-2} t^{d/2-2} \left[1 - \frac{4m^2}{s_{12}}\right]^{d/2-3/2} \left[(s - q^2)u - q^2 t\right]^{d/2}$$

Massive Wilson Coefficients – Pure-Singlet

Algorithm for the systematic integration:

- Perform the first 3 integrations in terms of polylogarithmic functions of involved arguments.
- Transform this integrand into independent iterated integrals with argument $\sim s_{12}$.
- Perform the last integration as iteration on top.

We find a large number of involved letters:

$$\frac{1}{1 \pm kt}, \frac{1}{1 \pm \beta t}, \frac{1}{k \pm z - (1-z)kt}, \frac{1}{k \pm z + (1-z)kt}, \frac{t}{k^2(1-t^2(1-z^2)) - z^2},$$
$$\frac{1}{t\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \frac{t}{k \pm z - (1-z)kt}, \frac{t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}(k^2(1-t^2(1-z^2)) - z^2)}$$

with $k = \frac{\sqrt{z}}{\sqrt{1-(1-z)\beta^2}}$.

Massive Wilson Coefficients – Pure-Singlet

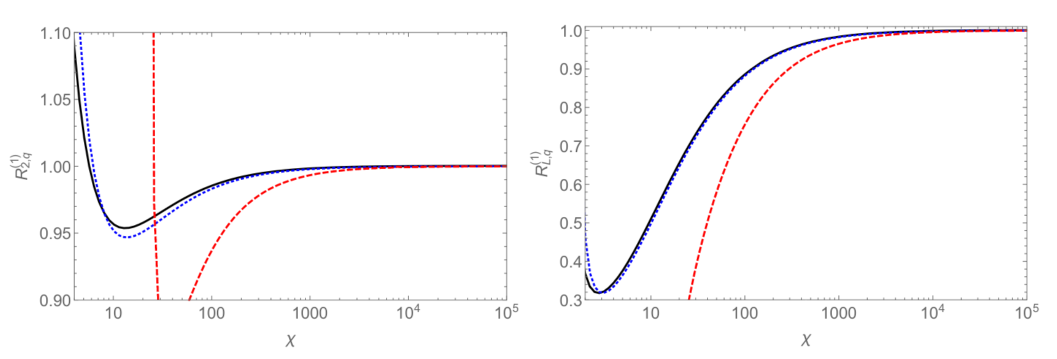
- This can be compared to the prediction of the asymptotic limit:

$$H_{L,q}^{(2),\mathrm{PS}}\left(z, \frac{Q^2}{m^2}\right) = \tilde{C}_{q,L}^{(2),\mathrm{PS}}(N_F+1)\ ,$$
$$H_{2,q}^{(2),\mathrm{PS}}\left(z, \frac{Q^2}{m^2}\right) = A_{Qq}^{(2),\mathrm{PS}}(N_F+1) + \tilde{C}_{q,2}^{(2),\mathrm{PS}}(N_F+1)$$

- The expression in terms of iterated integrals allows a systematic expansion in the asymptotic limit $Q^2 \gg m^2$.

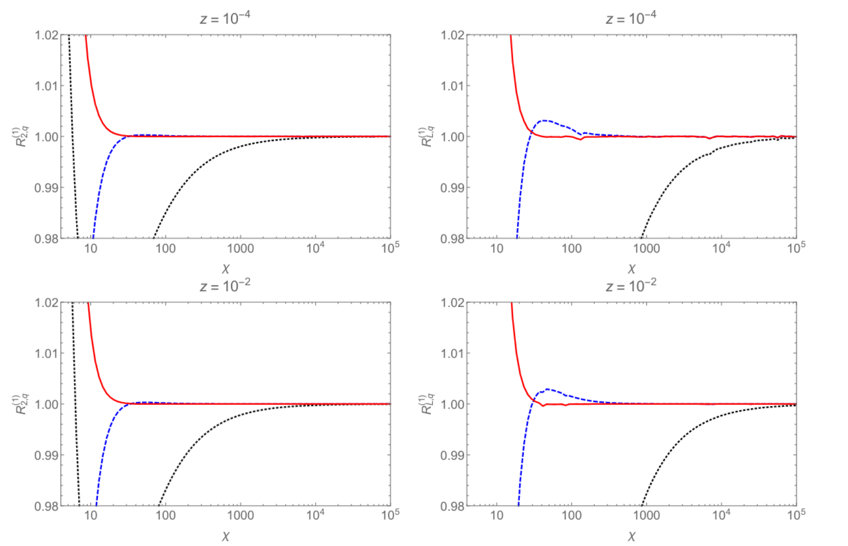
$$
\begin{aligned}
H_{L,q}^{2,\mathrm{PS}} \;=\; & -32 C_F T_F \Biggl\{ \frac{(1-z)\bigl(1-2z+10z^2\bigr)}{9z} - (1+z)(1-2z)\mathrm{H}_0 - z\mathrm{H}_0^2 \\
& + \frac{(1-z)\bigl(1-2z-2z^2\bigr)}{3z}\mathrm{H}_1 - z\mathrm{H}_{0,1} + z\zeta_2 + \frac{m^2}{Q^2}\Biggl[-\frac{(1-z)\bigl(2-z+2z^2\bigr)}{3z} \ln^2\left(\frac{m^2}{Q^2}\right) \\
& + \frac{(1-z)\bigl(-22+4z+29z^2\bigr)}{9z} - \Biggl(\frac{(1-z)\bigl(20-7z-25z^2\bigr)}{9z} + \frac{2}{3}(3-6z \\
& -2z^2)\mathrm{H}_0 \Biggr) \ln\left(\frac{m^2}{Q^2}\right) + \Biggl(\frac{2}{9}\bigl(-6+3z+13z^2\bigr) + \frac{2(1+z)\bigl(-2+z+2z^2+2z^3\bigr)}{3z} \\
& \times \mathrm{H}_{-1} \Biggr) \mathrm{H}_0 - \frac{2}{3} z^3 \mathrm{H}_0^2 + \Biggl(-\frac{(1-z)^2(14+13z)}{9z} + \frac{4(1-z)\bigl(2-z+2z^2\bigr)}{3z}\mathrm{H}_0 \Biggr) \mathrm{H}_1 \\
& + \frac{(1-z)\bigl(2-z+2z^2\bigr)}{3z}\mathrm{H}_1^2 - \frac{2\bigl(4-3z-4z^3\bigr)}{3z}\mathrm{H}_{0,1} \\
& + \frac{2(1+z)\bigl(2-z-2z^2-2z^3\bigr)}{3z}\mathrm{H}_{0,-1} - \frac{2(1-z)\bigl(2-z+2z^2+2z^3\bigr)}{3z}\zeta_2 \Biggr]
\end{aligned}
$$

Massive Wilson Coefficients – Pure-Singlet



The ratio of the full over the asymptotic results for $z = 10^{-4}, 10^{-2}, 1/2$.

Massive Wilson Coefficients – Pure-Singlet



The ratio of the full over the asymptotic results including terms of $\mathcal{O}\left((m^2/Q^2)^0\right)$, $\mathcal{O}\left((m^2/Q^2)^1\right)$, $\mathcal{O}\left((m^2/Q^2)^2\right)$.

Massive Operator Matrix Elements

Computing Massive Operator Matrix Elements

- We want to calculate massive operator matrix elements: $A_{ij} = \langle i \,| O_j |\, i \rangle$, with the operators

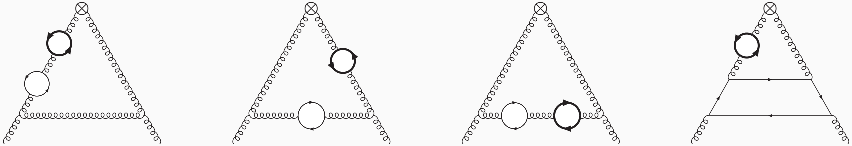
$$O^{\rm NS}_{q,r;\mu_1,\dots,\mu_N} = i^{N-1} \mathcal{S} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms} \,, \qquad (1)$$

$$O^{\rm S}_{q,r;\mu_1,\dots,\mu_N} = i^{N-1} \mathcal{S} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{trace terms} \,, \qquad (2)$$

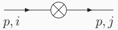
$$O^{\rm S}_{g,r;\mu_1,\dots,\mu_N} = 2 i^{N-2} \mathcal{S} \left[F^a_{\mu_1 \alpha} D_{\mu_2} \dots D_{\mu_N} F^{\alpha,a}_{\mu_N} \right] - \text{trace terms} \qquad (3)$$

and on-shell external partons $i = q, g$.

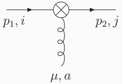
- The operator insertions introduce Feynman rules which depend on the Mellin variable N.



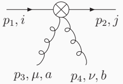
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



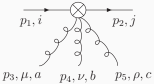
$$\delta^{ij}\not\Delta\gamma_\pm(\Delta\cdot p)^{N-1}\,,\quad N\geq 1$$



$$g t^a_{ji}\Delta^\mu\not\Delta\gamma_\pm\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\,,\quad N\geq 2$$



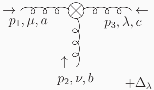
$$g^2\Delta^\mu\Delta^\nu\not\Delta\gamma_\pm\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}$$
$$\Big[(t^a t^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^b t^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\,,$$
$$N\geq 3$$



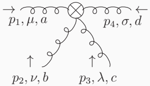
$$g^3\Delta_\mu\Delta_\nu\Delta_\rho\not\Delta\gamma_\pm\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}$$
$$\Big[\ (t^a t^b t^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^a t^c t^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^b t^a t^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^b t^c t^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}$$
$$+(t^c t^a t^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^c t^b t^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\,,$$
$$N\geq 4$$

$$\gamma_+=1\,,\quad \gamma_-=\gamma_5\,.$$

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\,,\quad N\geq 2$$



$$-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg($$
$$\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}$$
$$+\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu+\Delta\cdot p_2 p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]$$
$$\times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}$$
$$+\left\{{p_1\to p_2\to p_3\to p_1 \atop \mu\to\nu\to\lambda\to\mu}\right\}+\left\{{p_1\to p_3\to p_2\to p_1 \atop \mu\to\lambda\to\nu\to\mu}\right\}\Bigg)\,,\quad N\geq 2$$



$$g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$
$$+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\,,$$

$$O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}$$
$$+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4 g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}$$
$$-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}$$
$$+[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4 p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 p_{4,\mu}\Delta_\sigma]$$
$$\times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}$$
$$-\left\{{p_1\leftrightarrow p_2 \atop \mu\leftrightarrow\nu}\right\}-\left\{{p_3\leftrightarrow p_4 \atop \lambda\leftrightarrow\sigma}\right\}+\left\{{p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4 \atop \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma}\right\}\,,\quad N\geq 2$$

Status of the Operator Matrix Element

Leading Order: [Witten (1976); Babcock, Sivers, Wolfram (1978); Shifman, Vainshtein, Zakharov (1978); Leveille, Weiler (1979); Glück, Reya (1979); Glück, Hoffmann, Reya (1982)]

Next-to-Leading Order:

full m dependence (numeric) [Laenen, van Neerven, Riemersma, Smith (1993)]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven (1996)]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein (2007)]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein (2008, 2009)]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments (using `MATAD` [Steinhauser (2000)]):
  - F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein (2009)]
  - transversity: $N = 1...13$
  - Two masses $m_1 \neq m_2 \rightarrow$ Moments N = 2,4,6 [Blümlein, Wißbrock (2011)]
- Analytic solutions for $A_{qq,Q}^{\rm NS}$, $A_{qg,Q}$, $A_{gq,Q}$, $A_{qq,Q}^{\rm PS}$, $A_{Qq}^{\rm PS}$ [Blümlein et al (2010-2023)] , with recent extension to polarized scattering.
- Analytic two mass solutions for $A_{qq,Q}^{\rm NS}$, $A_{qg,Q}$, $A_{gq,Q}$, $A_{qq,Q}^{\rm PS}$, $A_{Qq}^{\rm PS}$, $A_{gg,Q}$ [Blümlein et al (2017-2020)] , with recent extension to polarized scattering.

The heavy flavor Wilson coefficients in the asymptotic limit:

$$
\begin{aligned}
L_{q,(2,L)}^{\rm NS}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]\\
L_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]\\
L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.\\
&\quad\left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]\\
H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]\\
&\quad + a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right]\\
H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]\\
&\quad + a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]\\
&\quad + a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.\\
&\quad\left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]
\end{aligned}
$$

Variable Flavor Number Scheme

Matching conditions for parton distribution functions:

$$f_k(N_F+1)+f_{\overline{k}}(N_F+1)=A_{qq,Q}^{\mathrm{NS}}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{\mathrm{PS}}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot G(N_F)\ ,$$

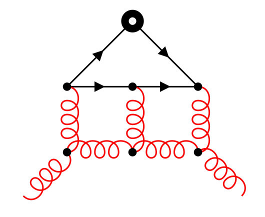
$$f_Q(N_F+1)+f_{\overline{Q}}(N_F+1)=A_{Qq}^{\mathrm{PS}}\left(N_F+1,\frac{m_1^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot G(N_F)\ ,$$

$$\Sigma(N_F+1)=\left[A_{qq,Q}^{\mathrm{NS}}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)+A_{qq,Q}^{\mathrm{PS}}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)+A_{Qq}^{\mathrm{PS}}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)+A_{Qg}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\right]\cdot G(N_F)\ ,$$

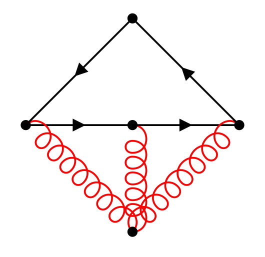
$$G(N_F+1)=A_{gq,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+1,\frac{m_1^2}{\mu^2}\right)\cdot G(N_F)\ .$$

Moments of the Massive Operator Matrix Elements

- For fixed N the operator reduced to a simple numerator.
- Since $p^2 = 0$ the propagators can be reduced to tadpole integrals.
- Up to three loop these calculations can e.g. be done with `MATAD` [Steinhauser '00] .



$\overset{p=0}{\Longrightarrow}$



$$\times \prod_{i=1}^{2} \frac{1}{l_i^2} \sum_{j=0}^{\infty} \left(\frac{2p \cdot l_i}{l_i^2} \right)^j$$

The Operator Matrix Element $A_{Qg}^{(3)}$

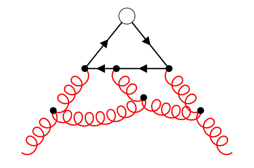
- The diagrams are given by propagators with operator insertions.
- Operators can be summed into propagator structures:

$$(\Delta.k)^N \to \sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t\ \Delta.k}$$

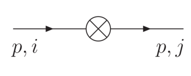
$$\sum_{j=0}^{N} (\Delta.k_1)^j (\Delta.k_2)^{N-j} \to \sum_{N=0}^{\infty} \sum_{j=0}^{N} t^N (\Delta.k_1)^j (\Delta.k_2)^{N-j}$$

$$= \frac{1}{[1 - t\ \Delta.k_1][1 - t\ \Delta.k_2]}$$

- With the linear propagators we can use IBP reductions.
- We can derive a system of differential equations in t.



Additional Feynman rules, e.g.:



$$\delta^{ij} \not\Delta \gamma_{\pm} (\Delta \cdot p)^{N-1}$$

The Operator Matrix Element $A_{Qg}^{(3)}$

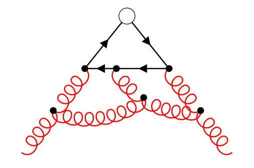
- The diagrams are given by propagators with operator insertions.
- Operators can be summed into propagator structures:

$$(\Delta.k)^N \to \sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t\ \Delta.k}$$

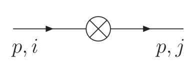
$$\sum_{j=0}^{N} (\Delta.k_1)^j (\Delta.k_2)^{N-j} \to \sum_{N=0}^{\infty} \sum_{j=0}^{N} t^N (\Delta.k_1)^j (\Delta.k_2)^{N-j}$$

$$= \frac{1}{[1 - t\ \Delta.k_1][1 - t\ \Delta.k_2]}$$

- With the linear propagators we can use IBP reductions.
- We can derive a system of differential equations in t.



Additional Feynman rules, e.g.:



$$\delta^{ij} \not\Delta \gamma_{\pm} (\Delta \cdot p)^{N-1}$$

# diagrams	# of masters	# of factorizing masters	# of factorizing diagrams
1233	666	468	1009

The Operator Matrix Element $A_{Qg}^{(3)}$ – Factorizable Part

- Alphabet for the massless Wilson coefficients (harmonic polylogarithms): [Remiddi, Vermaseren '99]

$$\left\{\frac{1}{t}, \frac{1}{1 \pm t}\right\}$$

- Alphabet for $A_{Qg}^{(3)}$:

$$\left\{\frac{1}{t}, \frac{1}{1 \pm t}, \frac{1}{2 \pm t}, \frac{1}{4 \pm t}, \frac{1}{1 \pm 2t}, \sqrt{t(4 \pm t)}, \frac{\sqrt{t(4 \pm t)}}{1 - t}, \frac{\sqrt{t(4 \pm t)}}{1 + t}, \frac{\sqrt{t(4 \pm t)}}{1 \mp 2t}\right\}$$

- Numerical solution we used expansions around: $t_0 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{4}{3}, 2, 4, \infty\right\}$
- Analytic and numeric solution agree on the level of 10^{-10} or better.

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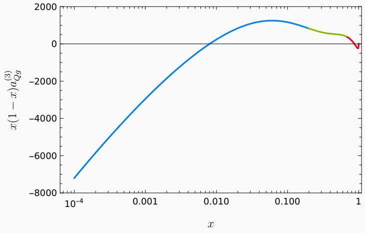

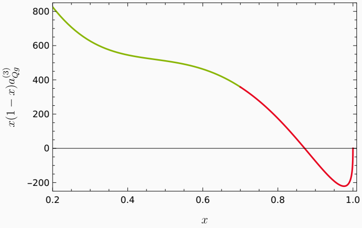

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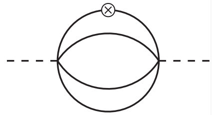
- Analytic and numeric solution agree on the level of 10^{-10} or better.

$\rightarrow$ Extension to the full problem will provide the last missing OME at $\mathcal{O}(\alpha_s^3)$.

The Operator Matrix Element $A_{Qg}^{(3)}$ – Non-Factorizable Part

$$\frac{d}{dt}\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & -\frac{1}{1-t} & 0 \\ 0 & -\frac{1}{t(1-t)} & -\frac{2}{1-t} \\ 0 & \frac{2}{t(8+t)} & \frac{1}{8+t} \end{bmatrix}\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} + \begin{bmatrix} R_1(t,\varepsilon) \\ R_2(t,\varepsilon) \\ R_3(t,\varepsilon) \end{bmatrix} + O(\varepsilon),$$



$$\begin{aligned} R_1(t,\varepsilon) &= \frac{1}{t(1-t)\varepsilon^3}\left[16 - \frac{68}{3}\varepsilon + \left(\frac{59}{3} + 6\zeta_2\right)\varepsilon^2 + \left(-\frac{65}{12} - \frac{17}{2}\zeta_2 + 2\zeta_3\right)\varepsilon^3\right] + O(\varepsilon), \\ R_2(t,\varepsilon) &= \frac{1}{t(1-t)\varepsilon^3}\left[8 - \frac{16}{3}\varepsilon + \left(\frac{4}{3} + 3\zeta_2\right)\varepsilon^2 + \left(\frac{14}{3} - 2\zeta_2 + \zeta_3\right)\varepsilon^3\right] + O(\varepsilon), \\ R_3(t,\varepsilon) &= \frac{1}{12t(8+t)\varepsilon^3}\left[-192 + 8\varepsilon - 8\left(4 + 9\zeta_2\right)\varepsilon^2 + \left(68 + 3\zeta_2 - 24\zeta_3\right)\varepsilon^3\right] + O(\varepsilon). \end{aligned}$$

The Operator Matrix Element $A_{Qg}^{(3)}$ – Non-Factorizable Part

After decoupling for $F_1(t)$ we find the differential equation

$$f_1^{(3)}(t) - \frac{2(4+5t)}{t(1-t)(8+t)} f_1^{(2)}(t) + \frac{4}{t(1-t)(8+t)} f_1^{(1)}(t) = 0$$

with $F_1(t) = f_1(t)/t$ and

We the methods of [Immamoglu, van Hoeij '17] implemented in `Maple` we find solutions for $f_1^{(1)}(t)$:

$$g_1(t) = \frac{t^2(8+t)^2}{(4-t)^4} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; z(t)\right],$$

$$g_2(t) = \frac{t^2(8+t)^2}{(4-t)^4} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; 1 - z(t)\right]$$

with

$$z(t) = \frac{27t^2}{(4-t)^3}$$

The Operator Matrix Element $A_{Qg}^{(3)}$ – Non-Factorizable Part

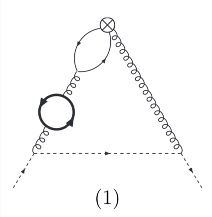
- When decoupling for F_3 first, we find:

$$F_1'(t) + \frac{1}{t} F_1(t) = 0, \quad g_0 = \frac{1}{t}$$

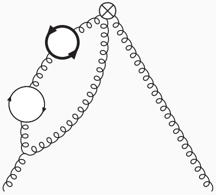
$$\boxed{F_3''(t) + \frac{(2-t)}{(1-t)t} F_3'(t) + \frac{2+t}{(1-t)t(8+t)} F_3(t) = 0,}$$

with

$$\begin{aligned}
g_1(t) &= \frac{2}{(1-t)^{2/3}(8+t)^{1/3}} \, {}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{4}{3} \\ 2\end{matrix}; -\frac{27t}{(1-t)^2(8+t)}\right], \\
g_2(t) &= \frac{9\sqrt{3}\Gamma^2(1/3)}{8\pi} \frac{1}{(1-t)^{2/3}(8+t)^{1/3}} \, {}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{4}{3} \\ \frac{2}{3}\end{matrix}; 1 + \frac{27t}{(1-t)^2(8+t)}\right], \\
W(t) &= \frac{1-t}{t^2}
\end{aligned}$$



$A_{qq}^{PS,(3)}$



$A_{qq}^{PS,(3)}$

- The mass ratio between charm and bottom quark is not so small:

$$\boxed{\frac{m_c^2}{m_b^2} \sim 0.1}$$

- We can go into the asymptotic limit $Q^2 \gg m_c^2, m_b^2$ without neglecting power corrections.
- We have to consider massive operator matrix elements with two internal heavy quark masses.
- Contributions due to two massive quarks are more involved due to an additional scale.
  $\Rightarrow$ Moments are already functions of η.

Massive Operator Matrix Elements – Two Mass Contributions

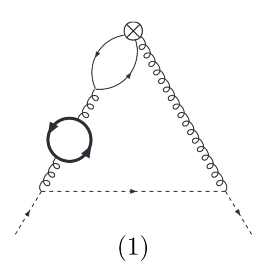
Contributions due to two massive quarks are more involved due to an additional scale.

- Problem: Mellin space expression cannot be found algorithmically.
- Solution: We used direct derivation of momentum space via Mellin-Barnes representations.
- We find integrals of the form:

$$J_1 = \int_0^1 \mathrm{d}x\, x^N \left\{ x^{\epsilon/2}(1-x)^{1+\epsilon/2} B_1\left(\frac{\eta}{x(1-x)}\right)\right\},$$

$$B_1(y) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}\sigma\, y^\sigma \Gamma(-\sigma)\Gamma(-\sigma+\epsilon)\Gamma(\sigma-\frac{3\epsilon}{2})\Gamma(\sigma-\frac{\epsilon}{2})\frac{\Gamma^2(\sigma+2-\epsilon)}{\Gamma(2\sigma+4-2\epsilon)}$$

- Depending on $y > 1$ or $y < 1$ close the contour to the left or right and sum residues.
- The residue sums can be rewritten in terms of iterated integrals (involving root valued letters).



$A_{qq}^{PS,(3)}$

Massive Operator Matrix Elements – Two Mass Contributions

Contributions due to two massive quarks are more involved due to an additional scale.

- Find Feynman parametrization:

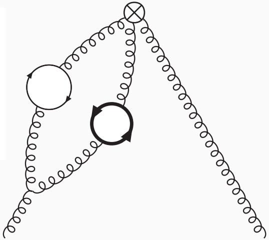
$$\begin{aligned} I \;=\; & \Gamma\left(-\tfrac{3\varepsilon}{2}\right) \int_0^1 \left(\prod_{i=1}^{7} \mathrm{d}z_i\right) z_1^2 \left(z_2(1-z_2)\right)^{\frac{\varepsilon}{2}} z_3^2 \left(z_4(1-z_4)\right)^{\frac{\varepsilon}{2}} (1-z_5)\left(z_6(1-z_6)\right)^{\frac{\varepsilon}{2}} z_7^{1+\frac{\varepsilon}{2}} \\ & \times (1-z_7)^2 \left(z_7(z_1 z_6 + z_3(1-z_6)) + z_5(1-z_7)\right)^{N-4} \left(\frac{z_6\, m_a^2}{z_2(1-z_2)} + \frac{(1-z_6)\, m_b^2}{z_4(1-z_4)}\right)^{\frac{3\varepsilon}{2}} \quad (4.8) \end{aligned}$$

- Split the masses by introducing a Mellin-Barnes intgeral:

$$\frac{1}{(A+B)^s} = \frac{1}{2\pi i}\frac{1}{\Gamma(s)} B^{-s} \int_{-i\infty}^{+i\infty} \mathrm{d}\sigma \left(\frac{A}{B}\right)^{\sigma} \Gamma(-\sigma)\Gamma(\sigma+s),$$

- Split the operator polynomial:

$$\left(z_7(z_1 z_6 + z_3(1-z_6)) + z_5(1-z_7)\right)^{N-4} = \sum_{j=0}^{N-4}\sum_{i=0}^{j} \binom{N-4}{j}\binom{j}{i} z_7^j z_1^i z_6^i z_3^{j-i} (1-z_6)^{j-i} z_5^{N-4-j} (1-z_7)^{N-4-j}$$



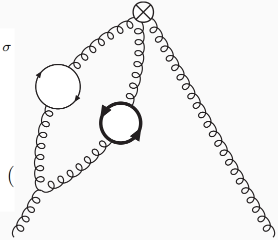
$A_{gg,Q}^{(3)}$

Massive Operator Matrix Elements – Two Mass Contributions

Contributions due to two massive quarks are more involved due to an additional scale.

- We find:

$$I = \frac{(m_b^2)^{\frac{3\varepsilon}{2}}}{2\pi i} \sum_{j=0}^{N-4} \sum_{i=0}^{j} \binom{N-4}{j} \binom{j}{i} \frac{\Gamma(3+i)\Gamma(3-i+j)\Gamma(N-j-3)}{\Gamma(4+i)\Gamma(4-i+j)\Gamma(N+1+\frac{\varepsilon}{2})} \int_{-i\infty}^{+i\infty} d\sigma \left(\frac{m_a^2}{m_b^2}\right)^{\sigma}$$
$$\times \Gamma(-\sigma)\Gamma(-\tfrac{3\varepsilon}{2}+\sigma)\Gamma(1-\tfrac{\varepsilon}{2}+i+\sigma)\Gamma(1+\varepsilon-i+j-\sigma)$$
$$\times \frac{\Gamma(1+\frac{\varepsilon}{2}-\sigma)\Gamma(3+\frac{\varepsilon}{2}-\sigma)\Gamma(1-\varepsilon-\sigma)\Gamma(3-\varepsilon+\sigma)}{\Gamma(4+\varepsilon-2\sigma)\Gamma(4-2\varepsilon+2\sigma)}.$$



$A_{gg,Q}^{(3)}$

- Take residue sums and obtain terms like:

$$T(\varepsilon,\eta,N) = \sum_{j=0}^{N} \sum_{i=0}^{j} \sum_{k=0}^{i} f(\varepsilon,\eta,N,j,i,k) = \sum_{j=0}^{N} \sum_{i=0}^{j} \sum_{k=0}^{i} \frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon} \times$$
$$\times\, 2^{-2+\varepsilon} e^{-\frac{3\varepsilon\gamma}{2}} \eta^k \frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}.$$

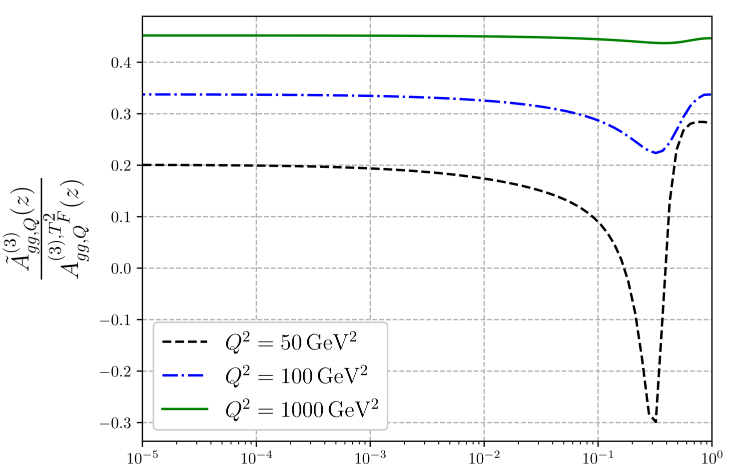
In Mellin space we find binomial sums:

$$\binom{2N}{N} \sum_{i=1}^{N} \frac{4^i \left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}$$

This translates into momentum space:

$$\int_0^x dz_1 \frac{\sqrt{z_1(1-z_1)}}{1-z_1(1-\eta)} \int_0^{z_1} dz_2 \frac{1}{1-z_2}$$

Massive Operator Matrix Elements – Two Mass Contributions



Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on ${}_2F_1$ functions

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx\ {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras

Except the last line integrals, all other ones stem from 1st order factorizable equations.

Thank you.

Backup

Other representations

- A similar solution was found for the analytic calculation of the ρ parameter at 3-loop order: [Ablinger, Blümlein, De Freitas, van Hoeij, Imamoglu '18]

$$\psi_{1a}^{(0)}(x) = \frac{x^2(x^2-1)(x^2-9)^2}{(x^2+3)^4}\,{}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{5}{3}\\ 2\end{matrix};\frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$
$$\sim -(x-1)(x-3)(x+3)^2\sqrt{\frac{x+1}{9-3x}}\,\mathrm{K}\left(-\frac{16x^3}{(x+1)(x-3)^3}\right)$$
$$+(x^2+3)(x-3)^2\sqrt{\frac{x+1}{9-3x}}\,\mathrm{E}\left(-\frac{16x^3}{(x+1)(x-3)^3}\right)$$

- In [Abreu, Becchetti, Duhr, Marzucca '22] it was shown that a representation in terms of eMPLs and iterated Eisenstein integrals exists.