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A few comments about the electroweak renormalisation

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The bare Lagrangian density

- The bare Lagrangian is a mathematical quantity,
  from which we derive the equations of motion of the fields and the scattering amplitudes

  It describes a k-fold infinite set of possible theories, parametric in the k masses and couplings
  (changing the value of the electric charge affects the chemistry but not the physics of our world!)

- A precise, physical, meaning of the Lagrangian is achieved imposing the renormalisation conditions

  The renormalisation program is not specifically related to the UV divergences,
  but it rather solves the k-fold infinite degeneracy of the bare lagrangian,
  choosing a specific value and meaning for the couplings

- The renormalisation conditions are imposed at a given energy scale, the renormalisation scale.
  The dependence of the theory on this choice is controlled by the Renormalization Group Equations

The couplings in the Lagrangian and their relation to physical quantities

- The EW SM is invariant under the gauge group $SU(2)_L \times U(1)_Y$, with gauge couplings g and g'
  The scalar sector depends on the VEV v of the Higgs field and on the quartic scalar coupling λ
  Neglecting the fermion masses and the fermion-scalar interaction, the theory is fully specified by 4 couplings

- In the construction of the SM there are two neutral currents
  we impose that one is the electromagnetic current, coupling to the photon field A^μ
  the second neutral current is in turn a prediction, coupling to the Z-boson field Z^μ
  The fields A^μ and Z^μ are a linear combination of the $SU(2)_L \times U(1)_Y$ gauge fields, with a rotation angle $\tan\theta_W = \dfrac{g'}{g}$
  and the electric charge is $e = g \sin\theta_W$

- After spontaneous symmetry breaking, the gauge bosons acquire mass, via the Higgs mechanism

$$m_W = \frac{1}{2} g v, \quad m_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}, \quad \text{while the Higgs boson mass is } m_H = v\sqrt{2\lambda}$$

- The Lagrangian couplings are in simple direct relation with four physical parameters

$$(g, g', v, \lambda) \quad \leftrightarrow \quad (e, m_W, m_Z, m_H)$$

(the weak mixing angle is a derived parameter)

A simple renormalisation scheme (Sirlin 1980)

- The tree level relations between the Lagrangian couplings and the chosen physical parameters hold for the bare quantities

$$e_0 = \frac{g_0\, g_0'}{\sqrt{g_0^2 + g_0'^2}}, \quad m_{W,0} = \frac{1}{2} g_0\, v_0, \quad m_{Z,0} = \frac{1}{2} v_0 \sqrt{g_0^2 + g_0'^2}, \quad m_{H,0} = v_0 \sqrt{2\lambda_0}$$

- We express the bare physical parameters in terms of renormalised ones and counterterms

$$e_0 = e + \delta e, \quad m_{W,0}^2 = m_W^2 + \delta m_W^2, \quad m_{Z,0}^2 = m_Z^2 + \delta m_Z^2, \quad m_{H,0}^2 = m_H^2 + \delta m_H^2$$

and also the same replacement for the Lagrangian couplings

$$g_0 = g + \delta g, \ g_0' = g' + \delta g', \ v_0 = v + \delta v, \ \lambda_0 = \lambda + \delta\lambda$$

- We expand in powers of $\hbar$ the relations and identify the coefficients, order by order

$$\delta a = (\hbar)^1 \delta a^{(1)} + (\hbar)^2 \delta a^{(2)} + (\hbar)^3 \delta a^{(3)} + \ldots$$

The Lagrangian coupling counterterms $(\delta g, \delta g', \delta v, \delta\lambda)$ can be expressed as a linear combination of $(\delta e, \delta m_W^2, \delta m_Z^2, \delta m_H^2,)$

- How can we compute $(\delta e, \delta m_W^2, \delta m_Z^2, \delta m_H^2)$?

Renormalisation conditions I

- The counterterm defines the renormalised parameter
  The relation between the renormalised parameter and the experimental input must then be specified

- In the on-shell renormalisation scheme, the renormalised mass coincides with the pole of the propagator

$$\frac{1}{p^2-m_0^2+\Sigma(p^2)} = \frac{1}{p^2-m^2-\delta m^2+\Sigma(p^2)} = \frac{1}{p^2-m^2-\delta m^2+\Sigma(m^2)+(p^2-m^2)\Sigma'(m^2)+\ldots} = \frac{1}{(p^2-m^2)\ \left(1+\Sigma'(m^2)\right)}$$

with $\delta m^2 = \Sigma(m^2)$

- In the on-shell renormalisation scheme, the request of probabilistic interpretation of the fields, leads to a condition on the residue of the propagator, which must be 1
  This leads to the definition of the renormalised fields $\phi_0 = \phi\, Z_{wf}^{\frac{1}{2}}$ with $Z_{wf} = 1 + \Sigma'(m^2)$

- These definitions stem from the study of the propagator and are completely general, for each field

Renormalisation conditions II

- The electric charge counterterm is defined via the study of the Thomson scattering
  i.e. the emission of a photon off a fermion, at vanishing momentum transfer

- In the on-shell renormalisation scheme, the electric charge counterterm is defined in such a way that the renormalised charge coincides with the experimental charge, at all orders in perturbation theory
  i.e. the counterterm cancels, order by order, the radiative corrections to the Thomson scattering,
  in that specific phase-space point

- The QED Ward Identities lead to an exact cancellation of vertex and fermionic WF corrections, so that the electric charge counterterm depends only on the external photon WF factor $\rightarrow$ universality of the electric charge

- The QED WIs can be restored in the full SM, by a convenient choice of the gauge fixing: the Background Field Gauge
  $\rightarrow$ also in the full SM the electric charge counterterm depends only on the external photon WF factor,
  which now includes also bosonic corrections

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The renormalised Lagrangian and the choice of the input parameters

- Once $(\delta e, \delta m_W^2, \delta m_Z^2, \delta m_H^2)$ have been computed from the relevant self-energies,
  the Lagrangian is completely assigned and expressed in terms of (e, m_W, m_Z, m_H)
  PS: the weak mixing angle is not another independent parameter

- Is this choice of input parameters unique? no: we could use also G_μ or $\sin^2\theta_{eff}^\ell$ replacing e or m_W

- How do we choose the input parameters?
  1) minimize the parametric uncertainty of the final results $\rightarrow$ $(\alpha, G_\mu, m_Z, m_H)$ are the best known quantities
  2) avoid the dependence on non-perturbative QCD uncertainties $\rightarrow (G_\mu, m_W, m_Z, m_H)$
  3) reabsorb in the definition of the input parameters large radiative corrections $\rightarrow (G_\mu, m_W, m_Z, m_H)$ makes it
  4) extract from the data the value of one input parameter via a fitting procedure
     $\rightarrow (G_\mu, m_W, m_Z, m_H)$ allows to fit m_W, instead $(G_\mu, \sin^2\theta_{eff}^\ell, m_Z, m_H)$ is needed to fit $\sin^2\theta_{eff}^\ell$

The Fermi constant and the parameterisation of the charged-current weak interaction

Fermi theory of β decay

muon decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$

$$\frac{1}{\tau_\mu} \to \Gamma_\mu \to G_\mu$$

QED corrections to Γ_μ necessary for precise determination of G_μ
computable in the Fermi theory (Kinoshita, Sirlin, 1959)

The independence of the QED corrections of the underlying model (Fermi theory vs SM) allows

- to define G_μ and to measure its value with high precision $G_\mu = 1.1663787(6)\ 10^{-5}\ \mathrm{GeV}^{-2}$

- to "reabsorb" in the G_μ definition the large logarithmic QED effects

The Fermi constant and the parameterisation of the charged-current weak interaction

- The Fermi theory and the SM can be identified, i.e. matched
  imposing that the muon decay amplitude at zero momentum transfer, in the two theories, coincide

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8m_W^2}(1+\Delta r)$$

  - the QED corrections, identical in both models, simplify
  - the non-QED corrections contribute to the definition of the matching at a given perturbative order, via Δr

- It is possible to compute Δr using (e, m_W, m_Z, m_H) as inputs in the on-shell scheme

$$\Delta r = \Delta\alpha(m_Z^2) - \frac{\cos^2\theta_W}{\sin^2\theta_W}\rho \; + \; \Delta r_{rem} \qquad \text{and approximately} \qquad \Delta r \sim 0.07 - 3 \cdot 0.01 + \mathcal{O}(0.001) \sim 0.035$$

- Δr is a finite physical correction.
  Its inclusion allows to use G_μ as input to express the strength of the weak interaction

Predictivity of the Standard Model

$$\mathcal{L}_{SM} = \mathcal{L}_{SM}(\alpha, G_\mu, m_Z; m_H; m_f; CKM)$$

We trade m_W for G_μ among the inputs, and solve the matching condition for m_W

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8 m_W^2}(1+\Delta r) \quad \Rightarrow \quad m_W^2 = \frac{m_Z^2}{2}\left(1+\sqrt{1-\frac{4\pi\alpha}{G_\mu\sqrt{2} m_Z^2}(1+\Delta r)}\right)$$

Extracting SM parameters from the kinematical distributions (e.g. at the LHC)

- The Drell-Yan kinematical distributions can be computed in terms of the Lagrangian input parameters

If we keep m_W among the parameters, then we can compute several times e.g. the $d\sigma/dp_\perp^\ell$ distributions, with different m_W values (i.e. we prepare the "templates") and we can test which one best fits to the data

This is not possible e.g. in the $(\alpha, G_\mu, m_Z, m_H)$ input-scheme choice, because in this case m_W is fixed, is a prediction

- The same argument must be applied to the extraction of the effective weak mixing angle:
  $\rightarrow$ we need to have $\sin^2\theta_{eff}^\ell$ among the inputs

The presence of $\sin^2\theta_{eff}^\ell$ among the inputs implies that we can renormalise this parameter in the SM

Applying an "on-shell" definition, we define the counterterm in such a way that, order by order in perturbation theory, the renormalised parameter coincides with the experimental value, defined exactly at $q^2 = m_Z^2$, like at LEP.

In practice, the counterterm subtracts systematically all the radiative corrections which contribute to the redefinition of the vector coupling of the Z boson to fermions

Observables and pseudo-observables: gauge invariance issues

- The quantisation of the gauge theory requires the introduction of a gauge fixing term
  which explicitly breaks the gauge invariance, leading to gauge dependent Green's functions

  The BRS symmetry, including the Faddeev-Popov ghosts, guarantees that the S-matrix elements (→ the xsecs)
  are gauge invariant

  What about the parameters, like masses and couplings ?

- The position of the pole of the propagator, that we interpret as mass of the particle, depends on the mass ct definition

  If $\delta m_Z^2 = \mathrm{Re}(\Sigma_{ZZ}(p^2 = m_Z^2))$, then, starting from 2-loop EW, gauge dependent terms contribute to the renormalised mass

  The request that the mass ct completely removes the self-energy contribution, $\delta\mu_Z^2 = \Sigma_{ZZ}(p^2 = \mu_Z^2)$,
  solves the gauge invariance problem to all orders

  The couplings of the bare Lagrangian are real valued, and so it is the bare mass
  The gauge boson self-energy is complex valued, because of several internal thresholds
  The mass ct must be complex valued as well, because it exactly subtracts the self-energy
  The renormalised mass is thus complex valued. $m_0^2 = \mu^2 + \delta\mu^2$

Observables and pseudo-observables: gauge invariance issues

- The fact that an internal parameter is gauge dependent does not spoil the property that the xsecs are gauge invariant
  but
  if we try to fit that parameter and give it a physical meaning, it better be gauge invariant!

  For this reason, the complex-mass scheme has become popular in the last 15 years, in EW physics, to describe unstable particles

- The complex mass has two real-valued parameters, that we interpret as mass and decay-width of the particle: $\mu = m - \frac{i}{2}\Gamma$
  only one of the two is a free input parameter, the mass;
  the second parameter, the decay width, has to be computed according to the rules of the theory

- The decay width can be derived, according to the optical theorem, from the imaginary part of the self-energy of the boson.
  For consistency, in a calculation at $N^K LO$, we need the imaginary part of a self-energy a (k+1) loops

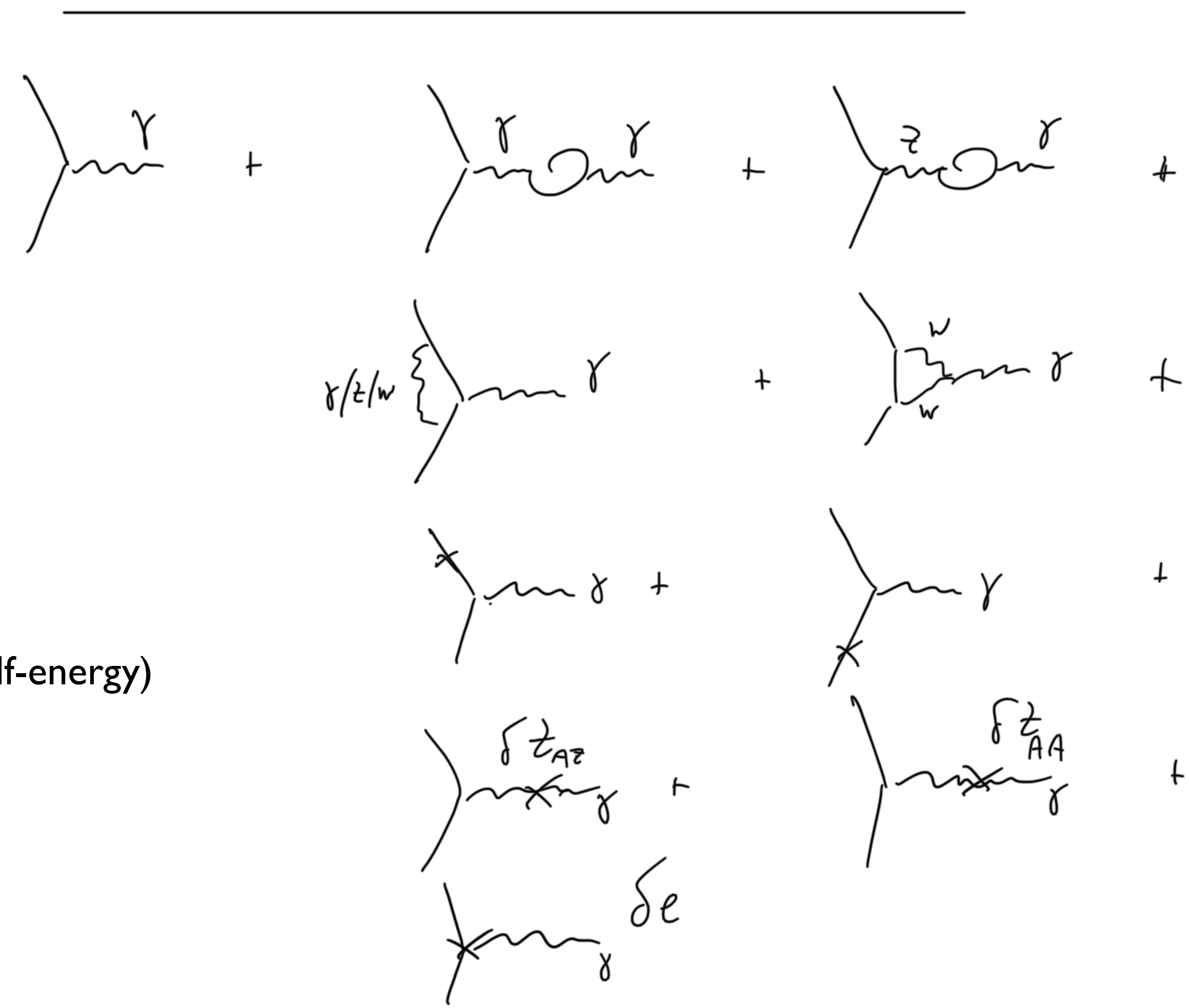
- All the Lagrangian bare parameters are real valued.
  The presence of complex-valued masses has a simple interpretation in the description of a resonance (position and width)
  The presence of complex-valued couplings instead should not be "dramatized": it is a rearrangement of the amplitude.
  E.g. a quantity defined in analogy to the weak mixing angle $s_w^2 = 1 - \frac{\mu_W^2}{\mu_Z^2}$ consistently develops an imaginary part,
  just as a factor which enters in the amplitude (Green's functions are in general complex valued)

The $\gamma f\bar{f}$ vertex

- $\gamma f\bar{f} \;\propto\; ie_0 Q_f \gamma^\mu$

In the $\gamma f\bar{f}$ vertex the $U(1)_{em}$ gauge symmetry protects
the magnetic form factor from getting renormalized

the electric charge CT cancel the UV divergences
stemming from the $\gamma\gamma$ self-energy
(in BGF, thanks to the restored transversality of the self-energy)



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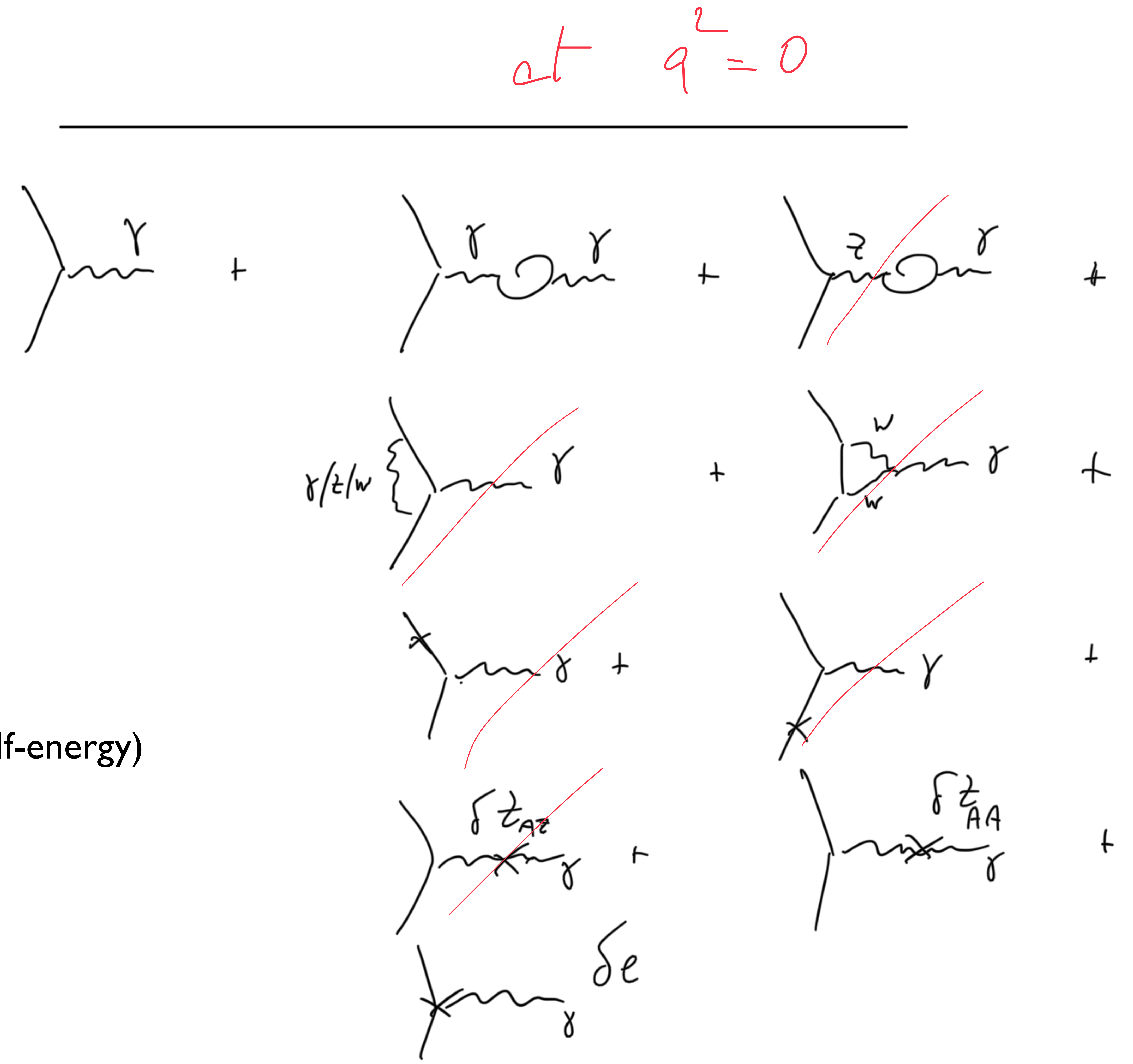

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The $Zf\bar{f}$ vertex

- $Zf\bar{f} \;\propto\; i\frac{g_0}{c_0}\gamma^\mu\left(T_3\frac{1-\gamma_5}{2} \;-\; s_0^2 Q_f\right)$

In the $Zf\bar{f}$ vertex we recognize the combination of the third weak isospin current with the e.m. current
the corresponding couplings get renormalized

the CTs of the overall coupling cancel the UV divergences stemming from the ZZ self-energy (in BGF)
→ the finite corrections contribute to the definition of the ρ parameter
the CTs of the s^2 in the e.m. current component cancel the UV divergences stemming from the γZ self-energy (in BGF)
→ the finite corrections partially contribute to the definition of an effective weak mixing angle via the κ parameter

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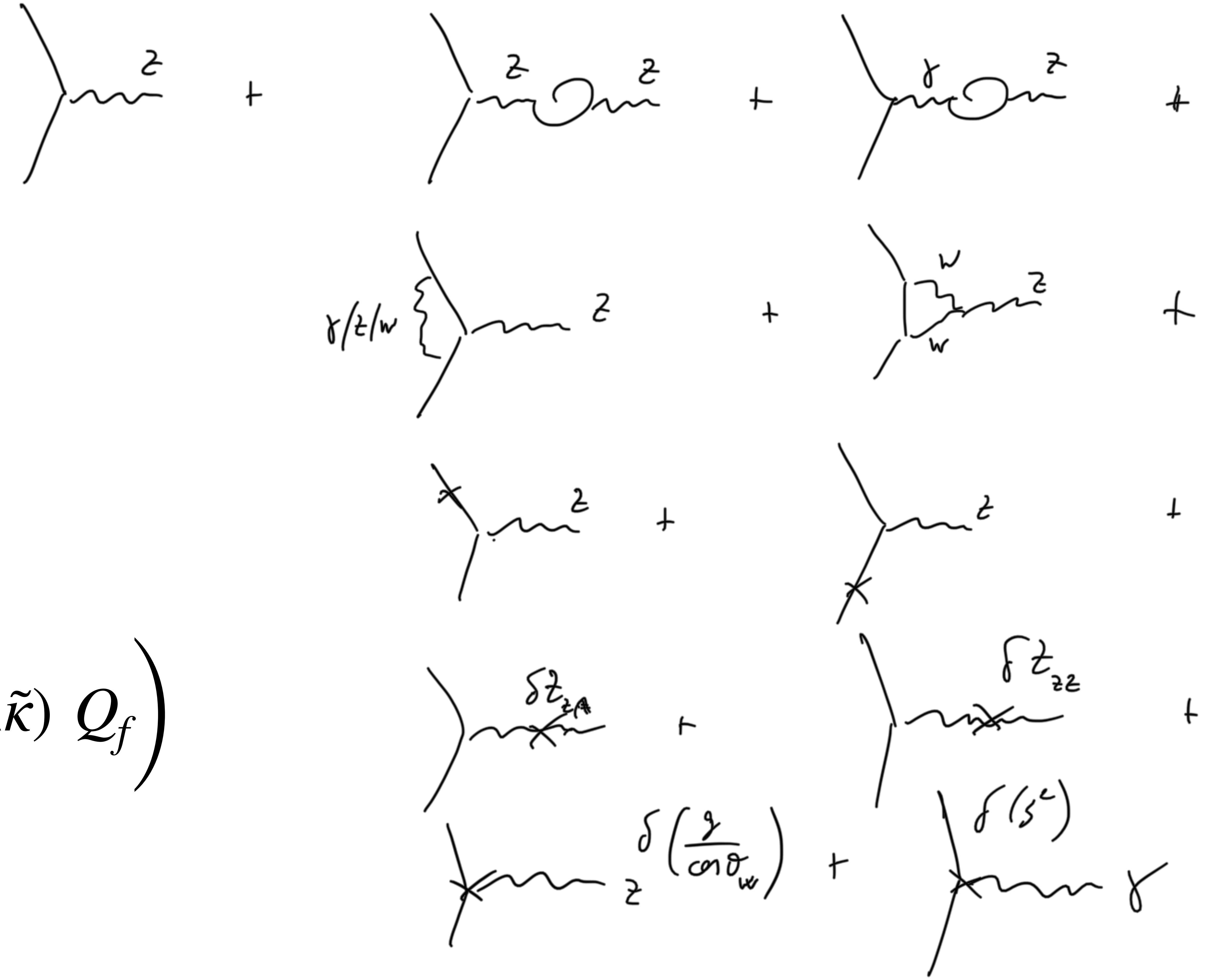
The $Zf\bar{f}$ vertex

- $Zf\bar{f} \;\propto\; i\dfrac{g_0}{c_0}\gamma^\mu\left(T_3\dfrac{1-\gamma_5}{2} \;-\; s_0^2 Q_f\right)$

$$\downarrow$$

$$i\frac{g}{c}(1+\Delta\tilde{\rho})\;\gamma^\mu\;\left(T_3\frac{1-\gamma_5}{2}\;-\;s^2\;(1+\Delta\tilde{\kappa})\;Q_f\right)$$



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Renormalised propagators

G.Degrassi, AV, hep-ph/0307122, S.Dittmaier,T.Schmidt,J.Schwarz, arXiv:2009.02229 S.Dittmaier, arXiv:2101.05154

Complex mass scheme

$$\mu_{W0}^2 = \mu_W^2 + \delta\mu_W^2, \quad \mu_{Z0}^2 = \mu_Z^2 + \delta\mu_Z^2, \quad e_0 = e + \delta e$$

$$\frac{\delta s^2}{s^2} = \frac{c^2}{s^2}\left(\frac{\delta\mu_Z^2}{\mu_Z^2} - \frac{\delta\mu_W^2}{\mu_W^2}\right)$$

the mass counterterms are defined at the complex pole of the propagator

the weak mixing angle is complex valued $c^2 \equiv \mu_W^2/\mu_Z^2$

BFG EW Ward identity → cancellation of the UV divergences combining vertex and fermion WF corrections

The bare couplings of Z and photon to fermions in the (G_μ, μ_W, μ_Z) input scheme are given by

$$\frac{g_0}{c_0} = \sqrt{4\sqrt{2}G_\mu\mu_Z^2}\left[1 - \frac{1}{2}\Delta r + \frac{1}{2}\left(2\frac{\delta e}{e} + \frac{s^2 - c^2}{c^2}\frac{\delta s^2}{s^2}\right)\right] \equiv \sqrt{4\sqrt{2}G_\mu\mu_Z^2}\left(1 + \delta g_Z^{G_\mu}\right)$$

$$g_0 s_0 = \sqrt{4\sqrt{2}G_\mu\mu_W^2 s^2}\left[1 + \tfrac{1}{2}\left(-\Delta r + 2\tfrac{\delta e}{e}\right)\right] \equiv e_{ren}^{G_\mu}\left(1 + \delta g_A^{G_\mu}\right)$$

Gauge boson renormalised propagators

$$\Sigma_{R,T}^{AA}(q^2) = \Sigma_T^{AA}(q^2) + 2\,q^2\,\delta g_A$$

$$\Sigma_{R,T}^{ZZ}(q^2) = \Sigma_T^{ZZ}(q^2) - \delta\mu_Z^2 + 2\,(q^2 - \mu_Z^2)\,\delta g_Z$$

$$\Sigma_{R,T}^{AZ}(q^2) = \Sigma_T^{AZ}(q^2) - q^2\,\frac{\delta s^2}{sc}$$

$$\Sigma_{R,T}^{ZA}(q^2) = \Sigma_T^{ZA}(q^2) - q^2\,\frac{\delta s^2}{sc},$$

γ_5 treatment in dimensional regularization

The absence of a consistent definition of γ_5 in $n = 4 - 2\varepsilon$ dimensions yields a practical problem

The trace of Dirac matrices and γ_5 is a polynomial in ε
The UV or IR divergences of Feynman integrals appear as poles $1/\varepsilon$

$$\mathrm{Tr}(\gamma_\alpha \ldots \gamma_\mu \gamma_5) \times \int d^n k \frac{1}{[k^2 - m_0^2][(k+q_1)^2 - m_1^2][(k+q_2)^2 - m_2^2]} \sim (a_0 + a_1 \varepsilon + \ldots) \times \left(\frac{c_{-2}}{\varepsilon^2} + \frac{c_{-1}}{\varepsilon} + c_0 + \ldots \right)$$

The evaluation of a_1 depends on the prescription adopted to handle γ_5 in $n = 4 - 2\varepsilon$ dimensions

Prescription-dependent poles appear in the evaluation of individual Feynman diagrams, as well as finite term.

Only a consistent treatment of all the contributions to the physical cross section leads to a cancellation of all the ε poles.
Additional care has to be devoted to the finite corrections.

γ_5 treatment in dimensional regularization

- Antisymmetric behavior under permutations is well defined in an integer number of dimensions.
  We can only specify the properties of γ_5 in the remaining $n-4$ dimensions with a prescription.

- The existence for some processes (e.g. $\Gamma(\pi_0 \to \gamma\gamma)$) of results evaluated with different regulators offers a benchmark

- 't Hooft-Veltman treat γ_5 (anti)commuting in (4) $n-4$ dimensions preserving the cyclicity of the traces
  (one counterterm is needed)

- Kreimer treats γ_5 anticommuting in n dimensions, abandoning the cyclicity of the traces ($\rightarrow$ need of a starting point)

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- We can classify amplitudes according to the presence/absence of closed fermionic triangles (anomalous contributions)

- Heller, von Manteuffel, Schabinger verified that in NNLO QCD-EW corrections to NC DY (no closed fermionic triangles) the UV-renormalized and IR-subtracted squared matrix element are identical in the two approaches

- The spurious, prescription dependent terms stem from the product of ε poles with γ-matrices traces
  The cancellation of the poles implies that of the spurious terms,
  provided that the amplitude and the subtraction term are computed in a fully consistent way

- The presence of closed fermionic triangles requires the usage of a prescription which correctly yields their finite part